

LYTLE

Double Limits

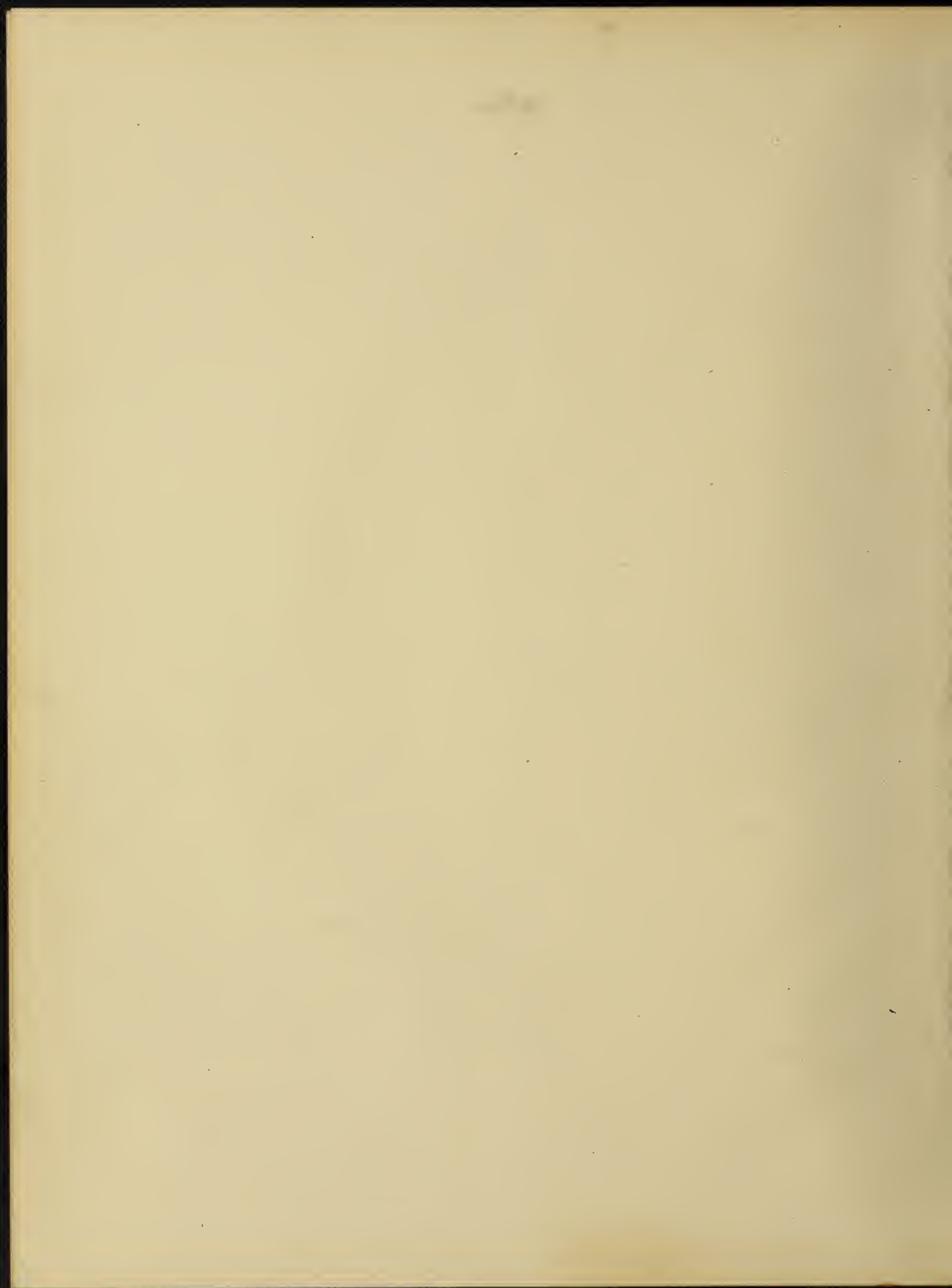
Mathematics and Physics

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DOUBLE LIMITS

by

Ernest B. Lytle.

Thesis

presented for the degree of

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in

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mr. Ernest B. Lytle.

ENTITLED

Double Limits

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF

Bachelor of Science (Math. and Physics)

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HEAD OF DEPARTMENT OF

Mathematics.

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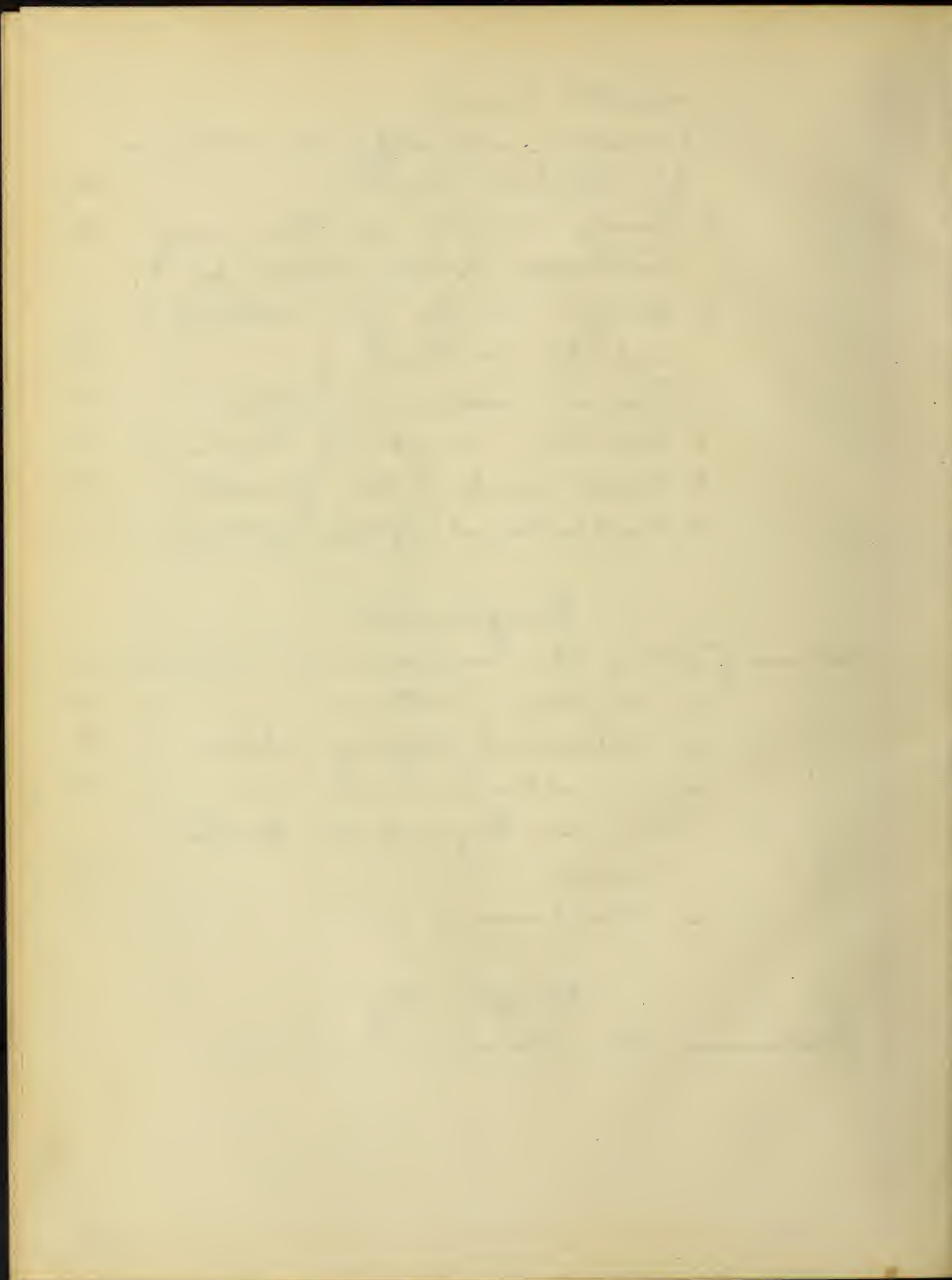
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Chapter I.

Preliminary Notions and Definitions.

1. We will use Dirichlet's definition of a function which is as follows: [See Harkness and Morley's Theory of Functions p. 53.] "Let x take certain values in an interval (x_0 to x_1); if y possess a definite value or definite values for each of these, y is said to be a function of x ."

We see that this definition does not require that y be related to x by any law or arithmetic expression. There are functions which cannot be given mathematical expression, such for example as Peano's problem. The failure to cover such cases is the weakness of the definition of function usually given. The above

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definition also has the advantage of covering such cases where x is defined for only certain values within the interval $(x_0 \text{ to } x_1)$; as when x is defined only for all rational numbers of the interval.

In this discussion we shall have under consideration functions of two real variables. Such a function, as $z = f(x, y)$, is completely defined when for every pair of values (x', y') which comes into consideration, there always exists a definite value of z . We shall restrict ourselves here to functions which are single valued; i. e. for every pair of values (x', y') there exists but a single value of z .

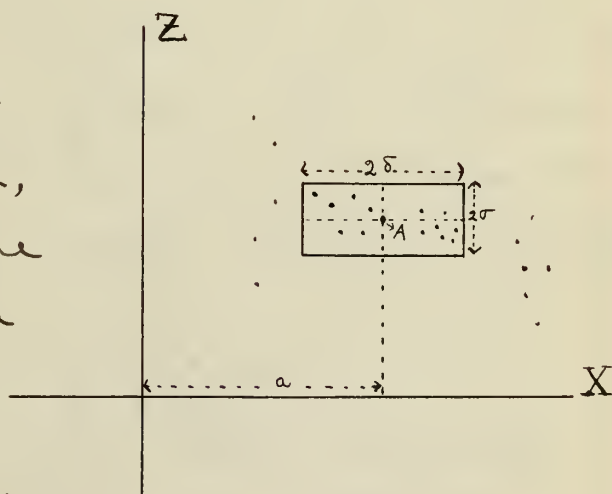
2. There may be two different kinds of variations of such functions of two variables; either we may consider one variable as constant and let the other vary by itself, or we may let both variables vary simultaneously.

These two methods of variation give rise to two kinds of limits. Of one variable, say y , be considered as constant, while x is made to approach the value a , then we say the single limit $\lim_{x \rightarrow a} f(x, b)$ exists and is equal to A , if for every arbitrarily small positive number σ there exists another positive number ϵ such that for every $|\delta| < \epsilon$ we have the relation

$$|f(a+\delta, b) - A| < \sigma$$

fulfilled.

The existence of this kind of a limit means geometrically that there is in the xz -plane around the point A a rectangle of 2σ in width, and 2ϵ in length, the ϵ depending upon the selection of the arbitrary σ and upon the point $x=a$, such that as x approaches the value a ,



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for every x between $a - \delta$ and $a + \delta$ the value of the function lies between $A + \sigma$ and $A - \sigma$ however small we choose the δ .

When x approaches the value a and y approaches the value b simultaneously, we get our second kind of limit which we will call a double limit and represent it by the symbol $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$.

We say that the double limit exists and is equal to A , when for every arbitrarily small positive number σ , there may be determined another positive number ϵ such that the relation

$$|f(a + \delta_1, b + \delta_2) - A| < \sigma \quad (1)$$

is true for every pair of values (δ_1, δ_2) where δ_1 and δ_2 are independent of one another and where furthermore

$$|\delta_1| \leq \epsilon, \quad |\delta_2| \leq \epsilon.$$

The existence of the double limit means geometrically that there is around the point A

My dear Mr. [Name] I have the honor to acknowledge the receipt of your letter of the 10th inst. in relation to the [subject] and in reply to inform you that the same has been forwarded to the proper authorities for their consideration. I am, Sir, very respectfully,
Yours obedient servant,
[Signature]

I am, Sir, very respectfully,
Yours obedient servant,
[Signature]

Very respectfully,
[Signature]

a parallelepiped 2σ by $2\delta_1$ by $2\delta_2$ in dimensions such that when

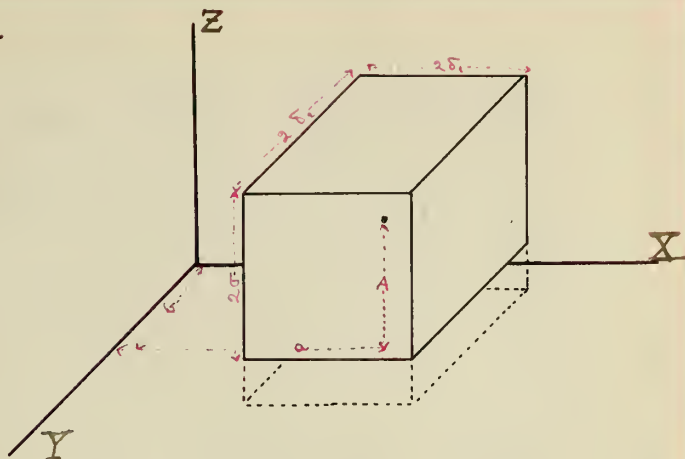
$$a - \delta_1 \leq x \leq a + \delta_1$$

$$b - \delta_2 \leq y \leq b + \delta_2$$

the value of the function (z) is always

$$A - \sigma \leq z \leq A + \sigma$$

however small we may choose the σ . Both δ_1 and δ_2 depend upon the values σ , a , and b .



The double limit may also be interpreted as follows:- If σ be any arbitrarily small positive number and if ϵ be any other positive number whose value depends upon σ and A , then the existence of the double limit means that there is around the point A a right circular cylinder, whose length is 2σ and whose base has a radius δ such that the value of the function always lies within this cylinder for every $|\delta| < \epsilon$, i.e. the value of the function is always

$$(A - \sigma) \leq Z \leq (A + \sigma)$$

Example 1. Given the function

$$Z = y \sin x.$$

Here we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} y \sin x = A = 0$$

for, however small we choose the σ , we can always find an ϵ such that for

$$|\delta_1| \leq \epsilon, \quad |\delta_2| \leq \epsilon$$

we have the relation

$$|\delta_2 \sin \delta_1| < \sigma$$

fulfilled.

The A which we have defined as the value of the double limit must always be a definite, finite number. Its value depends not upon the value of the function at the limiting point (a, b) but depends only upon the values of the function in the neighborhood of the limiting point (a, b) . If at the point (a, b) the double limit exists, its value need not be equal to the value of the function for $x=a, y=b$, i.e. $f(a, b)$; in fact

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The function may not even exist at the point (a, b) at all.

Example 2. Given the function

$$z = x^2 + y^2, \text{ where } f(0,0) = 1$$

When we define the function for $x=0, y=0$ as equal to one, we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} (x^2 + y^2) = 0 \neq f(0,0) = 1$$

i.e. the double limit exists but is not equal to the value of the function at the point $(0,0)$. Or if we consider our function as not defined for $x=0, y=0$, then we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} (x^2 + y^2) = 0.$$

3. If the double limit exists and is equal to the value of the function at the point (a, b) , i.e. if

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = f(a, b)$$

then we call the point (a, b) a regular point. At such regular points the function is always continuous with respect to both variables together, continuous with re-

The first part of the paper is devoted to a
discussion of the general principles of the
theory of the subject. It is shown that the
theory is based on the following assumptions:
(1) The system is in a state of equilibrium.
(2) The system is homogeneous.
(3) The system is isotropic.

The second part of the paper is devoted to a
discussion of the special case of the theory.
It is shown that the theory is based on the
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(1) The system is in a state of equilibrium.
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It is shown that the theory is based on the
following assumptions:
(1) The system is in a state of equilibrium.
(2) The system is homogeneous.
(3) The system is isotropic.

spect to each variable alone, and continuous by every continuous approach of x and y to such regular points. This will be made more clear in the following chapter.

If the double limit either does not exist or exists but is different from the value of the function for $x=a$, $y=b$, i.e. differs from $f(a,b)$, then we call the point an irregular point. We see at once that a function cannot be continuous with respect to both variables at irregular points.

4. At the point $x=a$, the magnitude of discontinuity, or sprung, of a function of a single variable is defined as the limit as $\delta \rightarrow 0$ of the difference between the upper and the lower limits of the function within the intervals $(a-\delta, a+\delta)$. In a similar way we define the sprung of a function of two variables $f(x,y)$ at the point (a,b) . Let us draw around the point (a,b) a circle with radius ρ and

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take the difference of the upper and the lower limits of the function within this circle. Then the limit as $\rho \rightarrow 0$ of this difference is the sprung of the function in respect to both variables, or in other words is simply the xy-sprung. It should be noticed that we take the difference between the upper and the lower limits and not the difference between the maximum and the minimum. This includes cases where there is no greatest or least values and therefore the use of upper and lower limits gives more generality than maximum and minimum would give.

At a regular point the sprung is always 0, for by definition the function is always continuous at regular points, and for continuous points of a function the amount of discontinuity or sprung must necessarily be 0. Since irregular points are points of discontinuity the sprung at such

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points must always be greater than 0.

Example 3. Given the function

$$z = \frac{xy}{x^2 + y^2}, \text{ where } f(0,0) = 0$$

After transforming to polar coordinates by substituting $x = \rho \cos \phi$, $y = \rho \sin \phi$, we see that the function has an upper limit when $\phi = 45^\circ$, and a lower limit when $\phi = -45^\circ$. The sprung at point $x=0, y=0$, is

$$\lim_{\rho \rightarrow 0} \left\{ \frac{\rho^2 \sin 45^\circ \cos 45^\circ}{\rho^2 \cos^2 45^\circ + \rho^2 \sin^2 45^\circ} - \frac{-\rho^2 \cos 45^\circ \sin 45^\circ}{\rho^2 \cos^2 45^\circ + \rho^2 \sin^2 45^\circ} \right\}$$

$$= 2 \sin 45^\circ \cos 45^\circ = \sin 90^\circ = 1.$$

5. When the double limit exists and is equal to the value of the function at that point, i. e. when

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x,y) = f(a,b)$$

then we call the function regularly convergent at the point (a,b) . In such cases the point (a,b) is a regular point. We always speak of a function being regularly convergent at a point and not for an interval. We shall see later that uniform convergence refers to an interval.

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6. If we pass through the surface $z = f(x, y)$ any plane parallel to either the zx -plane or the zy -plane then the curve of intersection we call an approximation curve. (Annäherungskurve). For example if we intersect the surface $z = f(x, y)$ by the plane $y = y_0$ then the curve $z = f(x, y_0)$ is an approximation curve. We will give some drawings of approximation curves in the last chapter.

7. We say that a function $f(x, y)$ converges uniformly toward $f(x, y_0)$ within the interval $a \leq x \leq b$ if, as y approaches y_0 simultaneously for all values of x between a and b , we reach the limiting value $f(x, y_0)$. This requires more than that

$$\lim_{y \rightarrow y_0} f(x_0, y) = f(x_0, y_0) \quad (1)$$

for every constant $x = x_0$ within the interval, which is only saying that $f(x_0, y)$ is continuous at the point $y = y_0$. Therefore we see that uniform convergence for the interval $a \leq x \leq b$ requires that, for an arbitrarily small

positive number σ and for every y_2 between $y_0 + \sigma$ and y_0 , the relation

$$|f(x, y_2) - f(x, y_0)| < \sigma \quad (2)$$

be fulfilled simultaneously for all values of x . This last relation (2) includes the whole interval $a \leq x \leq b$ while relation (1) has to do with but a single point of the interval. The single limit (1) means geometrically that there is a rectangle $f(x_0, y_0) \pm \sigma$ wide by $y_0 + \sigma$ long within which the function $f(x_0, y_2)$ must always lie, for $y_0 < y_2 \leq y_0 + \sigma$, $a \leq x_0 \leq b$ and where σ is any arbitrarily small positive number. Thus δ is a function of σ and x . When we consider the whole interval as in (2) then we may take our δ as having the same value for all x 's of the interval, provided we first select our σ . In such cases we may consider the δ as a function of σ alone.

If we project all the approximation curves upon the plane $y = y_0$ and draw on this plane a

strip of any arbitrarily small width σ on each side of the curve $Z = f(x, y_0)$, then after finding a y_2 such that the projection of $Z = f(x, y_2)$ lies wholly within this strip 2σ in width, the projection of every approximation curve between $Z = f(x, y_2)$ and $Z = f(x, y_0)$ must lie wholly within the strip if $f(x, y)$ converges uniformly toward $f(x, y_0)$.

8. It is our purpose to confine ourselves in this discussion strictly to the theory of double limits. But it may be of interest in this connection to suggest some applications of double limits without attempting their development. The theory of double limits finds an application in such questions as: - (1) Interchange of order of integration and differentiation, (2) Integration and differentiation of infinite series term by term, (3) Differentiation under the integral sign, (4) Condition for uniform convergence.

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9. In this chapter we have given some fundamental ideas and definitions. Chapter II will consist of theorems relating to the theory and properties of double limits. Chapter III will discuss methods of testing functions for the existence of double limits. The last chapter will be devoted to the discussion of special problems. There an attempt will be made to collect and discuss functions of two real variables which have any interest from the standpoint of double limits. Models of some surfaces there discussed will be constructed and drawings of some approximation curves will be made.

10. We are indebted to Prof. E. J. Townsend's Göttingen thesis, "Ueber den Begriff und die Anwendung des Doppellimes", for most of our theory of double limits, and also for several interesting functions. Paul du Bois-Reymond's article on "Theorie

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der Functionen zweier Veränderlichen," in Journal für die reine und angewandte Mathematik. Vol. 70 page 10, has given us some of the functions discussed in the last chapter. The writer is responsible for the translations, the arrangement of matter, working up of details, discussion of functions and for the construction of three models of surfaces discussed. A number of functions used in this thesis are original with the writer.

Chapter II.

Properties of Double Limits.

1. In this chapter we will give the properties of double limits in the form of propositions which we will demonstrate.

Proposition I:— If the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ exists and is equal

to A , then we must by every continuous approach of x and y to the point (a, b) obtain one and the same limiting value A ; for example, if we put $y = \phi(x)$ where $\phi(x)$ at the point $x = a$ is a continuous function and for $x = a$ has the value b , then we must have the relation

$$\lim_{x \rightarrow a} f(x, \phi(x)) = A.$$

Proof:— Suppose this were not true,

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then according to our definition of double limit, we could so select a function of x [$\phi(x)$] that for a sufficiently small σ and for every δ however small, we could have the relation

$$|f(a+\delta, \phi(a+\delta)) - A| > \sigma \quad (1)$$

But since $\phi(x)$ is continuous at the point $x=a$ and at this point has the value b , we have

$$\lim_{x \rightarrow a} \phi(x) = \phi(a) = b$$

or, what is the same thing, we have

$$\phi(a+\delta_1) = \phi(a) + \delta_2 = b + \delta_2$$

Substituting this value of $\phi(a+\delta_1)$ in

(1) we get

$$|f(a+\delta_1, b+\delta_2) - A| > \sigma$$

which is true for every pair of values (δ_1, δ_2) within a sufficiently small neighborhood of the point (a, b) . But this says that A is not the value of the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ and leads to a contradiction

of our assumed hypothesis. Therefore, our supposition that the proposition was not true is false and

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our proposition is established.

In a similar way it may be shown that

$$\lim_{y \rightarrow b} f(\psi(y), y) = A$$

if

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = A$$

and if $x = \psi(y)$ at the point $y = b$ is continuous and has the value a .

An important application of this proposition will be given in first section of the next chapter. This proposition shows us at once that a single valued function can have but one double limit at the same point.

2. Proposition II:- If the double limit exists and is equal to A , then

$$\lim_{x \rightarrow a} f(x, b) = \lim_{y \rightarrow b} f(a, y) = A.$$

This follows directly from proposition I, where we select our $\phi(x)$ as the constant b , or $\psi(y)$ as the constant a .

However, we must be careful

not to assume the converse; i.e. if the limits $\lim_{x \rightarrow a} f(x, \phi(x))$ and $\lim_{y \rightarrow b} f(\psi(y), y)$ both exist and are both equal to A , that the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$

also exists. This is not true as is shown by the following example.

Example 1. Given the function $Z = \frac{xy^3}{x^2 + y^6}$, where $f(0, 0) = 0$.

Here we have for $\phi(x) = 0$, and $\psi(y) = 0$

$$\lim_{x \rightarrow 0} f(x, 0) = \lim_{y \rightarrow 0} f(0, y) = 0.$$

But if we put $x = \psi(y) = y^3$, we have

$$\lim_{y \rightarrow 0} \frac{y^6}{y^6 + y^6} = \frac{1}{2}$$

and therefore the double limit $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$

does not exist by proposition I.

3. Proposition III:- If by every continuous simultaneous approach of x and y to the point (a, b) we come to the same limiting value A , then the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$

exists.

Proof. If this were not true then it would be possible to select some $\phi(x)$ [or $\psi(y)$], say $y = \phi'$, such

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that the limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ would either not exist or be different from A . Then by proposition I the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ would not exist. Therefore our proposition is true.

4. Proposition IV:- The necessary and sufficient condition that the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ exist is that

by every continuous approach of x and y to the point (a, b) , we always obtain the same limiting value A .

This is nothing more than a combination of propositions I and III and needs no new demonstration. This property makes our idea of the double limit more definite.

Still it gives us no way of testing for the existence of double limits, for it is impossible to find the limit for every continuous approach to the point (a, b) since there are an infinite number of them.

5. Proposition V:- The existence of the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ does

not require that the limits $\lim_{x \rightarrow a} f(x, b \pm \epsilon)$ and $\lim_{y \rightarrow b} f(a \pm \delta, y)$ exist, where ϵ and δ are arbitrarily small positive numbers.

Proof. This is a negative proposition and we will demonstrate it by showing a case where the double limit does exist and still the single limit $\lim_{x \rightarrow a} f(x, b \pm \epsilon)$ does not exist. Take the function

$$z = y \sin \frac{1}{x}, \text{ where } f(0,0) = 0, -1 < x < +1.$$

Here we see the double limit

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = 0.$$

while the limit

$$\lim_{x \rightarrow 0} f(x, 0 \pm \epsilon) = \lim_{x \rightarrow 0} \epsilon \sin \frac{1}{x} = \epsilon \sin \frac{1}{0}$$

which last has no interpretation.

Therefore the limit $\lim_{x \rightarrow 0} \epsilon \sin \frac{1}{x}$ does not exist and our proposition is demonstrated.

6. Proposition VI:- The existence of the limit $\lim_{x \rightarrow a} f(x, y)$ for every constant y , and the limit $\lim_{y \rightarrow b} f(x, y)$ for

every constant x , in the neighborhood of (a, b) , does not require that the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ exist.

Proof. Like the preceding proposition this is negative and we will demonstrate by means of an example. Given the function

$$z = \frac{2x + 3y}{x + y}, \text{ where } f(0, 0) = 0.$$

Here we have

$$\text{for } y \neq 0, \lim_{x \rightarrow 0} \frac{2x + 3y}{x + y} = 3;$$

$$" \quad y = 0, \lim_{x \rightarrow 0} f(x, y) = 2;$$

$$" \quad x = 0, \lim_{y \rightarrow 0} f(x, y) = 3;$$

$$" \quad x \neq 0, \lim_{y \rightarrow 0} f(x, y) = 2;$$

that is, all the single limits exist, yet, by proposition I, the double limit $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$ does not exist for

$$\lim_{x \rightarrow 0} f(x, 0) \neq \lim_{y \rightarrow 0} f(0, y).$$

Therefore our proposition is true.

Another similar example is

$$z = \left(\frac{y^2}{x^2 + y^2} \right)^{\frac{1}{x^2 + y^2}}, \text{ where } f(0, 0) = 0.$$

7. Proposition VII:- If the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ exists and is equal to

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A , and besides, if $\lim_{x \rightarrow a} f(x, y)$ for every y and also $\lim_{y \rightarrow b} f(x, y)$ for every x in the neighborhood of the point (a, b) exists, then

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = \lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y) = \lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y).$$

Proof. If the limit

$$\lim_{x \rightarrow a} f(x, y) = F(a, y)$$

then we have the relation

$$|F(a, y) - f(a + \delta, y)| < \sigma \quad (1)$$

for every y in the neighborhood of the point (a, b) . Since the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ exists and is equal

to A , we have, by the definition of double limit, the relation

$$|f(a + \delta, y) - A| < \sigma \quad (2)$$

for every δ and every y in the neighborhood of (a, b) . By adding (1) and (2) we get

$$|F(a, y) - A| < 2\sigma$$

or expressed in another form,

$$\lim_{y \rightarrow b} F(a, y) = A.$$

But

$$F(a, y) = \lim_{x \rightarrow a} f(x, y)$$

therefore, substituting we have

$$\lim_{y \neq b} \lim_{x \neq a} f(x, y) = A.$$

In a similar way we could show that

$$\lim_{x \neq a} \lim_{y \neq b} f(x, y) = A$$

therefore

$$\lim_{x \neq a} \lim_{y \neq b} f(x, y) = \lim_{y \neq b} \lim_{x \neq a} f(x, y) = A = \lim_{\substack{x \neq a \\ y \neq b}} f(x, y)$$

which proves our proposition.

This proposition does not say however that $F(a, y)$ shall be equal to $f(a, y)$. In other words, outside of the point (a, b) itself, the function $f(x, y)$ at no place need necessarily be continuous in respect to each variable alone, nor in respect to both variables.

We read the symbol $\lim_{\substack{x \neq a \\ y \neq b}} f(x, y)$

the double limit of $f(x, y)$ in respect to x and y ; the symbol $\lim_{x \neq a} \lim_{y \neq b} f(x, y)$ we call the twice taken limit of $f(x, y)$ in respect first to y and then to x .

8. Proposition VIII:- If the same double limit of two functions exists, the double limit of their sum is equal to

the sum of the double limits; i.e.,
if

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = A, \quad \text{and} \quad \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y) = B$$

then

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) \pm \phi(x, y)] = A \pm B.$$

Proof. By our definition of double limit we have

$$f(x, y) = A + \sigma_1 \quad (1.)$$

$$\phi(x, y) = B + \sigma_2 \quad (2.)$$

where both σ 's may be made as small as we please by taking x sufficiently near a and y sufficiently near b .
Adding (1) and (2) we have

$$f(x, y) \pm \phi(x, y) = A \pm B + (\sigma_1 \pm \sigma_2).$$

As $x \rightarrow a$ and $y \rightarrow b$ simultaneously the quantity $(\sigma_1 \pm \sigma_2)$ approaches zero. Therefore

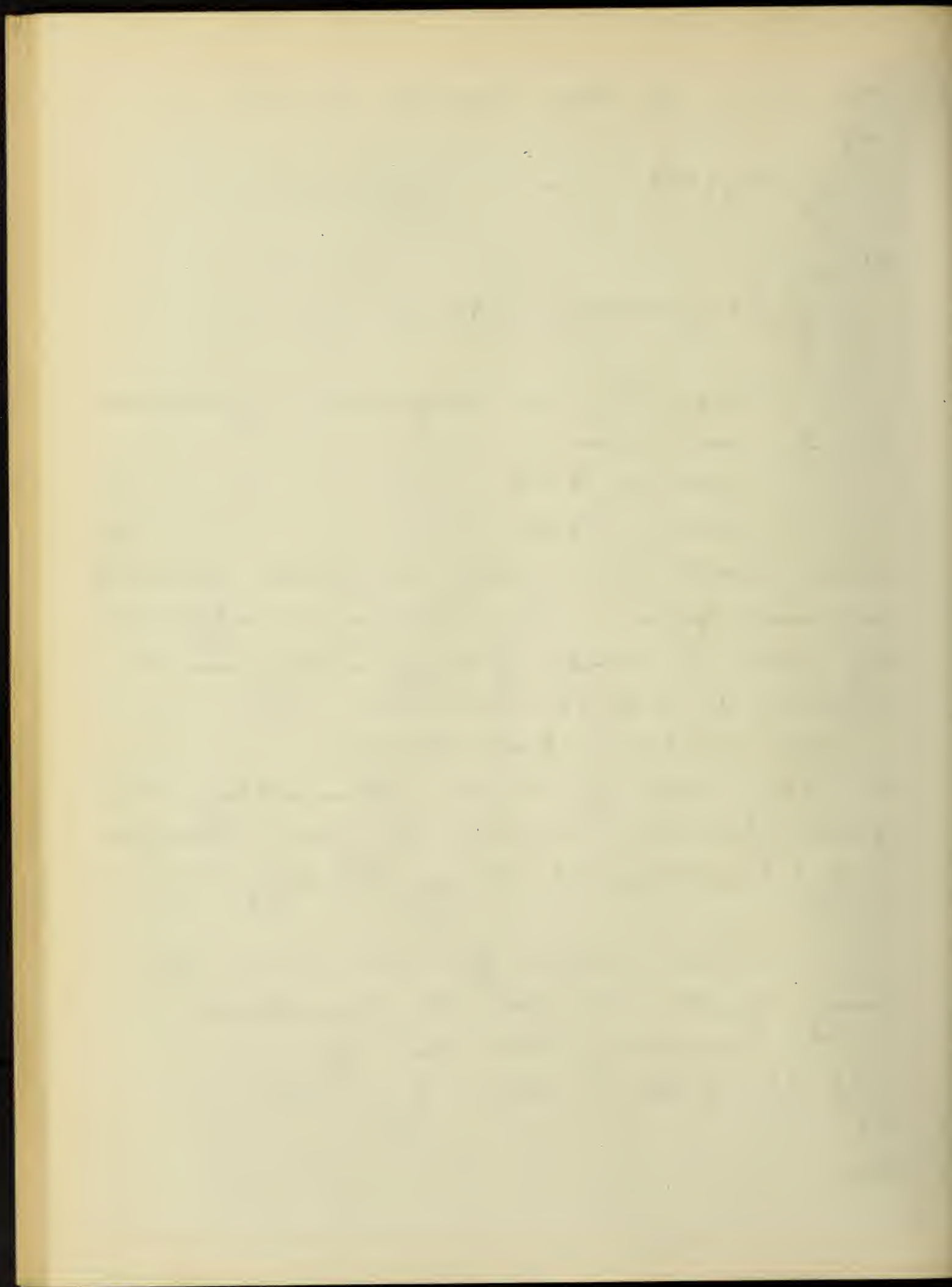
$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) \pm \phi(x, y)] = A \pm B = \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) \pm \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y).$$

This holds for the sum of any finite number of functions.

Corollary. We see if

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} (f + \phi) = A + B, \quad \text{and} \quad \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f = A$$

then



$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi = B.$$

For we may write

$$f + \phi = (A + B) + \sigma_3$$

$$f = A + \sigma_4$$

and subtracting we get

$$\phi = B + (\sigma_3 - \sigma_4)$$

or

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi = B,$$

since $(\sigma_3 - \sigma_4) \rightarrow 0$ as $x \rightarrow a$ and $y \rightarrow b$.

But we must not assume the converse of this proposition for $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \{f(x,y) + \phi(x,y)\}$ may exist without $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x,y)$

and $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x,y)$ exist, as shown by the

following example.

Example 2:- The double limits

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{2xy}{x^2 + y^2}$$

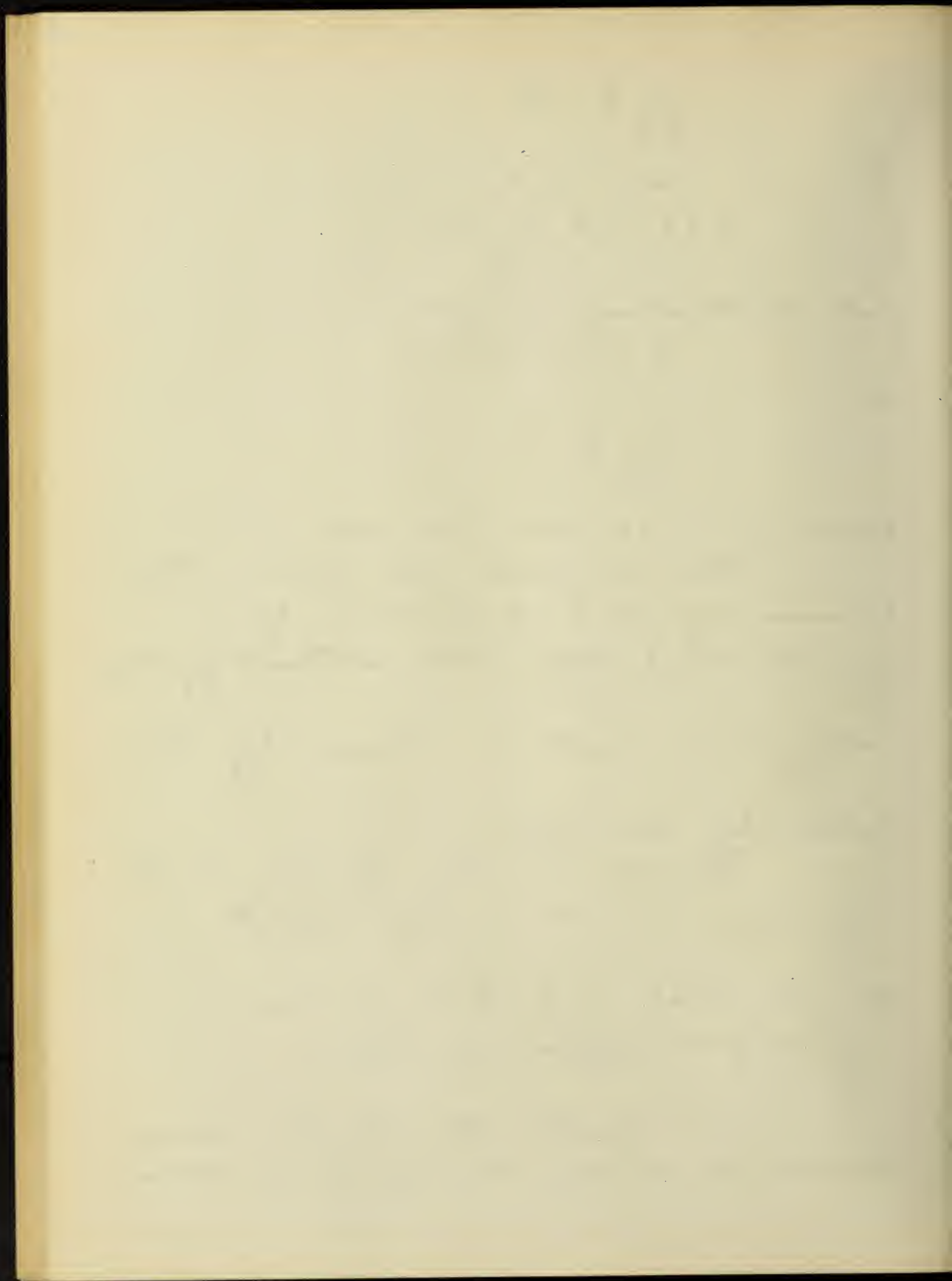
and

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 - 2xy + y^2}{x^2 + y^2}$$

do not exist, but still we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left\{ \frac{2xy}{x^2 + y^2} + \frac{x^2 - 2xy + y^2}{x^2 + y^2} \right\} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 + y^2}{x^2 + y^2} = 1.$$

9. Proposition IX:- If the same double limit of two functions exists,



the double limit of their product is equal to the product of their double limits; i.e. if

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = A, \text{ and } \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y) = B$$

then

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) \cdot \phi(x, y)] = A \cdot B.$$

Proof. By the definition of the double limit we have

$$f(x, y) = A + \sigma_1 \quad (1)$$

$$\phi(x, y) = B + \sigma_2 \quad (2)$$

where both σ 's approach zero as $x \rightarrow a$, $y \rightarrow b$ simultaneously. Taking the product of (1.) and (2.) we have

$$f(x, y) \cdot \phi(x, y) = AB + (A\sigma_2 + B\sigma_1 + \sigma_1\sigma_2).$$

But the quantity in parenthesis will approach zero as $x \rightarrow a$, $y \rightarrow b$. Therefore we may write

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) \cdot \phi(x, y)] = AB = \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) \cdot \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y).$$

Corollary. In the case of n equal factors we have

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \{f(x, y)\}^n = \left\{ \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) \right\}^n$$

if $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ exists and n is finite.

[The text on this page is extremely faint and illegible. It appears to be a list or a series of entries, possibly organized in columns, but the specific words and numbers cannot be discerned.]

The converse of this proposition is not true, as is shown by the following example.

Example 3:- The double limits

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{xy}{x^2+y^2}, \text{ and } \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2+y^2}{xy}$$

do not exist, but still we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{xy}{x^2+y^2} \right) \left(\frac{x^2+y^2}{xy} \right) = 1.$$

It should be noted that A.B can never have the indeterminate form $0 \cdot \infty$, since by definition of the double limit both A and B must be finite.

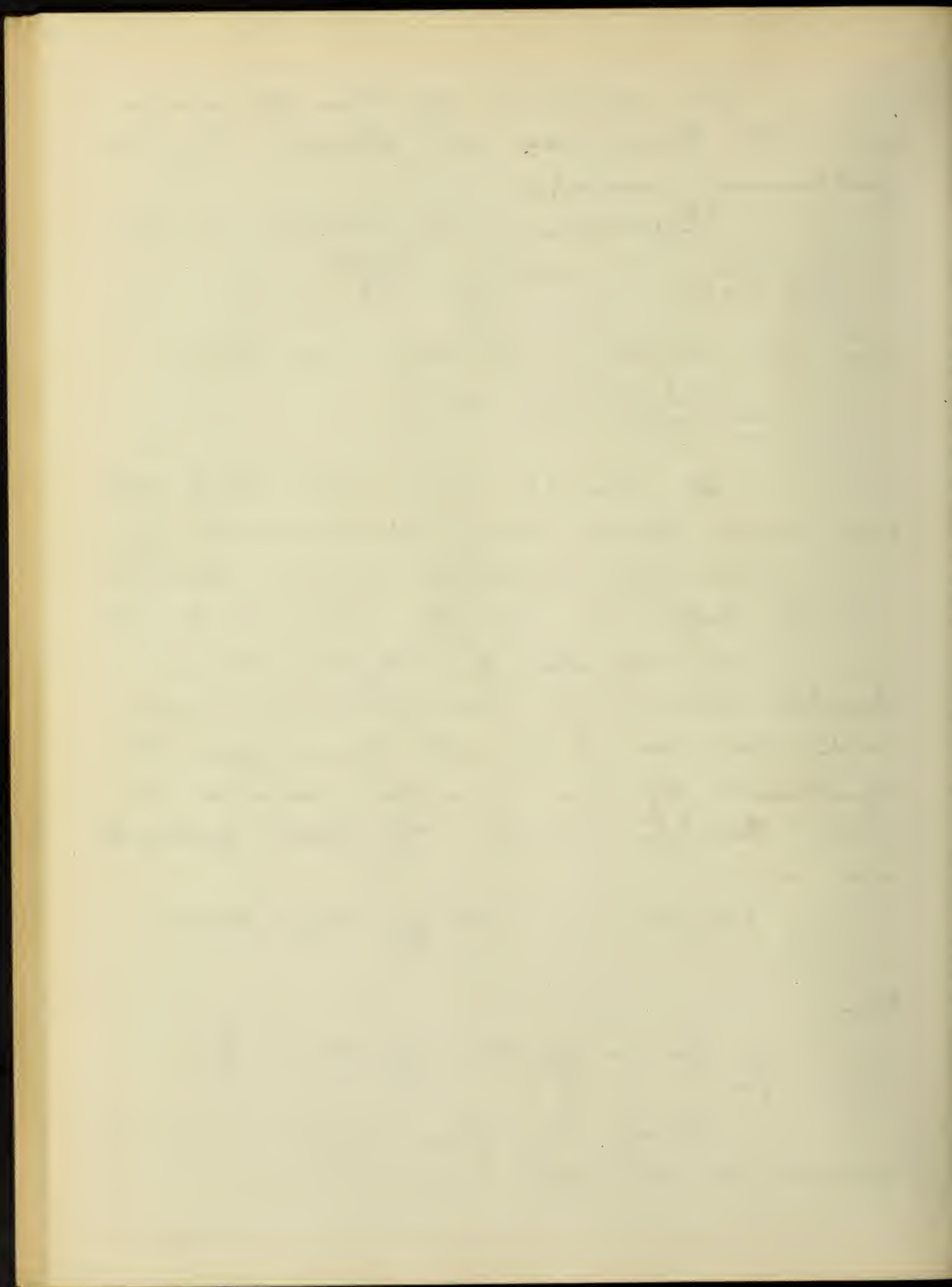
10. Proposition X:- If the same double limit of two functions exists and one be different from zero, the quotient of the double limits is the double limit of their quotient; i.e. if

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x,y) = A, \text{ and } \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x,y) = B \neq 0$$

then

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \frac{f}{\phi} = \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x,y) \div \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x,y) = \frac{A}{B}.$$

Proof. By the definition of the double limit we have



$$f(x, y) = A + \sigma_1 \quad (1.)$$

$$\phi(x, y) = B + \sigma_2 \quad (2.)$$

where the σ 's approach zero as $x \rightarrow a, y \rightarrow b$.

By division we have

$$\frac{f(x, y)}{\phi(x, y)} = \frac{A + \sigma_1}{B + \sigma_2} = \frac{A}{B + \sigma_2} + \frac{\sigma_1}{B + \sigma_2} \quad (3.)$$

Now as $x \rightarrow a, y \rightarrow b$ simultaneously the last member of (3.) approaches $\frac{A}{B}$. Therefore we can write,

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \frac{f(x, y)}{\phi(x, y)} = \frac{A}{B} = \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) \div \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y).$$

However the double limit

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \frac{f}{\phi} \text{ may exist without } \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) \text{ and}$$

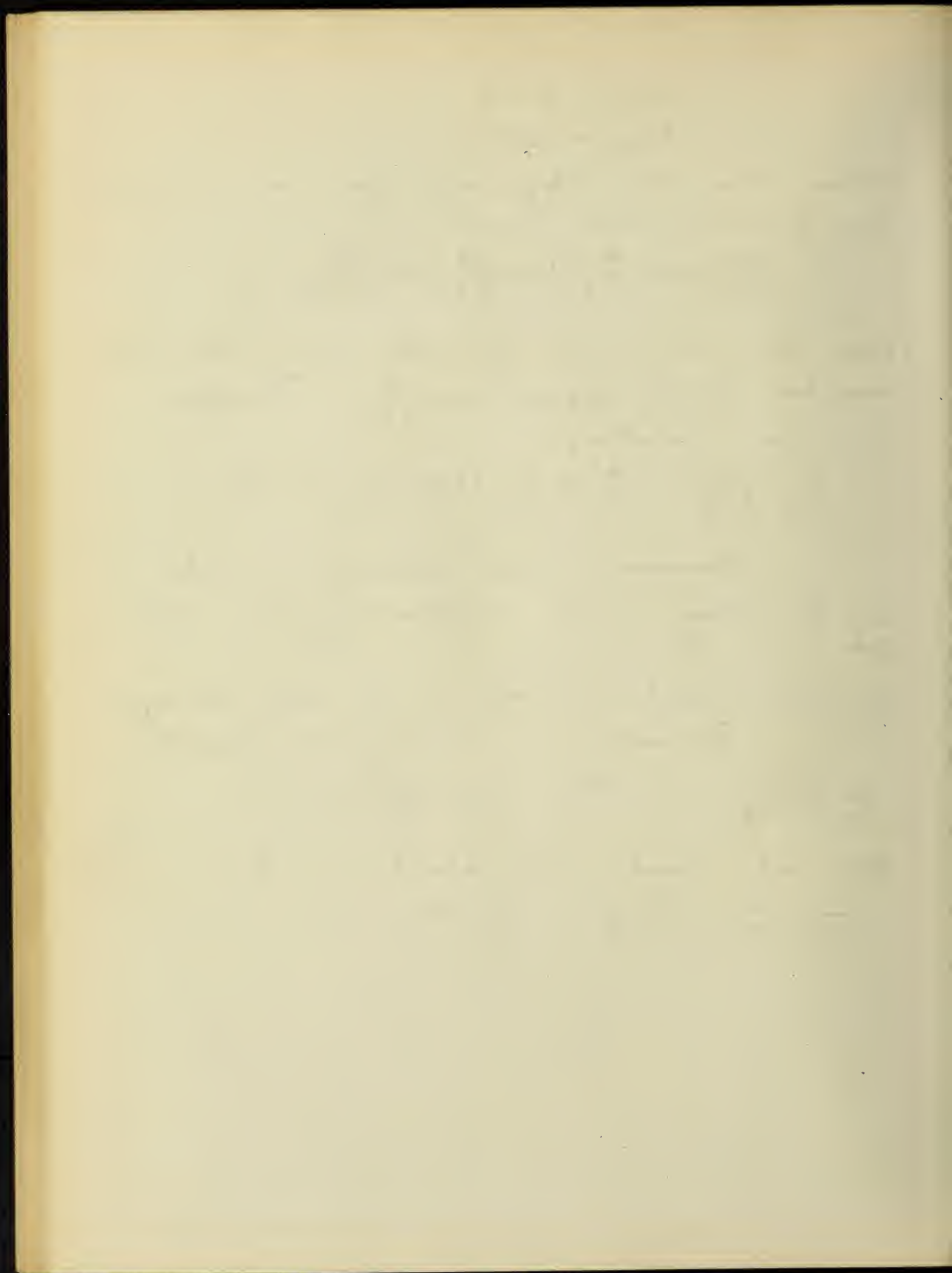
$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y)$ exist, as shown by this example.

Example 4 :- The double limits

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{xy^3}{x^2 + y^6} \right), \text{ and } \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(\frac{y^2}{x^2 + y^6} \right)$$

do not exist, but still we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left\{ \frac{xy^3}{x^2 + y^6} \div \frac{y^2}{x^2 + y^6} \right\} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} xy = 0.$$



Chapter III.

Methods of Testing the Existence of Double Limits.

1. In this chapter we expect to discuss methods of testing special functions for the existence of double limits.

In very many cases it is very easy to settle the question of the existence of the double limit by showing that the double limit does not exist. Proposition I of last chapter gives us a very practical negative test of this kind. There we saw that, if the double limit $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y)$ exists and is

equal to A , then

$$\lim_{x \rightarrow a} f(x, \phi(x)) = A$$

must hold for every $\phi(x)$ which is continuous and at the point $x = a$ has

the value b . Now if we can select our $\phi(x)$ in two ways such that $\lim_{x \rightarrow a} f(x, \phi(x))$ has two different limits then we know that the double limit does not exist at the point (a, b) .

Example 1:- Given the function

$$Z = \frac{x^2 - y^2}{x^2 + y^2}$$

to test at the point $x=0, y=0$, for the existence of double limit.

Let $y = 3x$ and we have

$$\lim_{x \rightarrow 0} \frac{x^2 - 9x^2}{x^2 + 9x^2} = -\frac{8}{10} = -\frac{4}{5}.$$

Let $y = 2x$ and we get

$$\lim_{x \rightarrow 0} \frac{x^2 - 4x^2}{x^2 + 4x^2} = -\frac{3}{5}$$

Therefore by proposition one of last chapter, we know that the double limit

$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

2. In some simple functions the defining relation of the first chapter gives us a positive test for the existence of the double limit. Suppose we put $y = x$ in $f(x, y)$, and find that

$$\lim_{x \rightarrow a} f(x, x) = A.$$

Then we know by Proposition I that, if there be a double limit at the point

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(a, b) , it must be A . Then if A is found to satisfy the defining relation

$$|f(a+\delta_1, b+\delta_2) - A| < \sigma \quad (1)$$

where $|\delta_1| < \delta$ and $|\delta_2| < \delta$, then we know that the double limit at the point (a, b) exists and is equal to A . However unless $f(x, y)$ is a rather simple function we have difficulty in determining whether A satisfies relation (1).

Example 2:- Given the function

$$z = y \tan x$$

to test for the existence of the double limit at point $x=0, y=0$.

Let $y=x$, and we have

$$\lim_{x \rightarrow 0} x \tan x = 0.$$

Then 0 must be the double limit if there be any. Substituting 0 in relation (1) we get

$$|(0+\delta_2) \tan(0+\delta_1) - 0| < \sigma$$

or

$$|\delta_2 \tan \delta_1| < \sigma.$$

Now by mere inspection we see that this relation is fulfilled, i.e. for every arbitrarily small positive number σ we can find a δ_1 and a δ_2 which

* Show if $f(p) = \sum_{n=0}^{\infty} a_n p^n$ is a function of p in the complex plane, then $f(p) = \sum_{n=0}^{\infty} a_n p^n$ for all p in the complex plane.

where $|a_n| < M$ for all n and p is a complex number.

Also $\sum_{n=0}^{\infty} |a_n| < \infty$ then the series of $f(p)$ converges uniformly.

satisfy this relation.

3. The use of polar coordinates is sometimes useful in testing for double limits. Suppose we intersect the surface $z = f(x, y)$ with a right circular cylinder of radius ρ whose axis coincides with the z -axis.

Then the curve of intersection is

$$z = f(\rho \cos \phi, \rho \sin \phi).$$

Since the double limit must by definition be a definite number then

$$\lim_{\rho \rightarrow 0} f(\rho \cos \phi, \rho \sin \phi)$$

must be finite and by Proposition I it must be independent of ϕ . Therefore we know if the

$$\lim_{\rho \rightarrow 0} f(\rho \cos \phi, \rho \sin \phi)$$

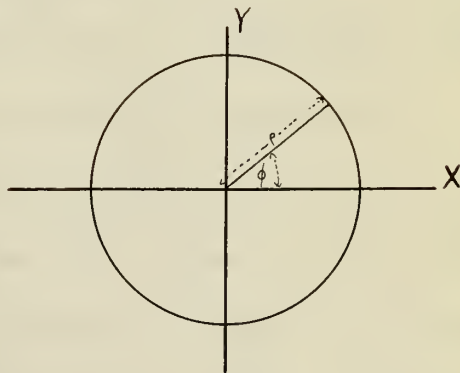
is not a definite finite number independent of ϕ

that the double limit at the point

at the point $(0, 0)$ does not exist. But

we must be careful not to assume that the double limit exists and is equal to A if

$$\lim_{\rho \rightarrow 0} f(\rho \cos \phi, \rho \sin \phi) = A^*$$



where A is a definite finite number.

Example 3:- Given the function

$$Z = \frac{xy^2}{x^2+y^4}.$$

Substituting $x = \rho \cos \phi$, $y = \rho \sin \phi$, we have

$$\lim_{\rho \neq 0} \frac{\rho \cos \phi \cdot \rho^2 \sin^2 \phi}{\rho^2 \cos^2 \phi + \rho^4 \sin^4 \phi} = \lim_{\rho \neq 0} \frac{\rho \cos \phi \sin^2 \phi}{\cos^2 \phi + \rho^2 \sin^4 \phi} = 0.$$

Still the double limit does not exist,

for put $x = my^2$ and we get

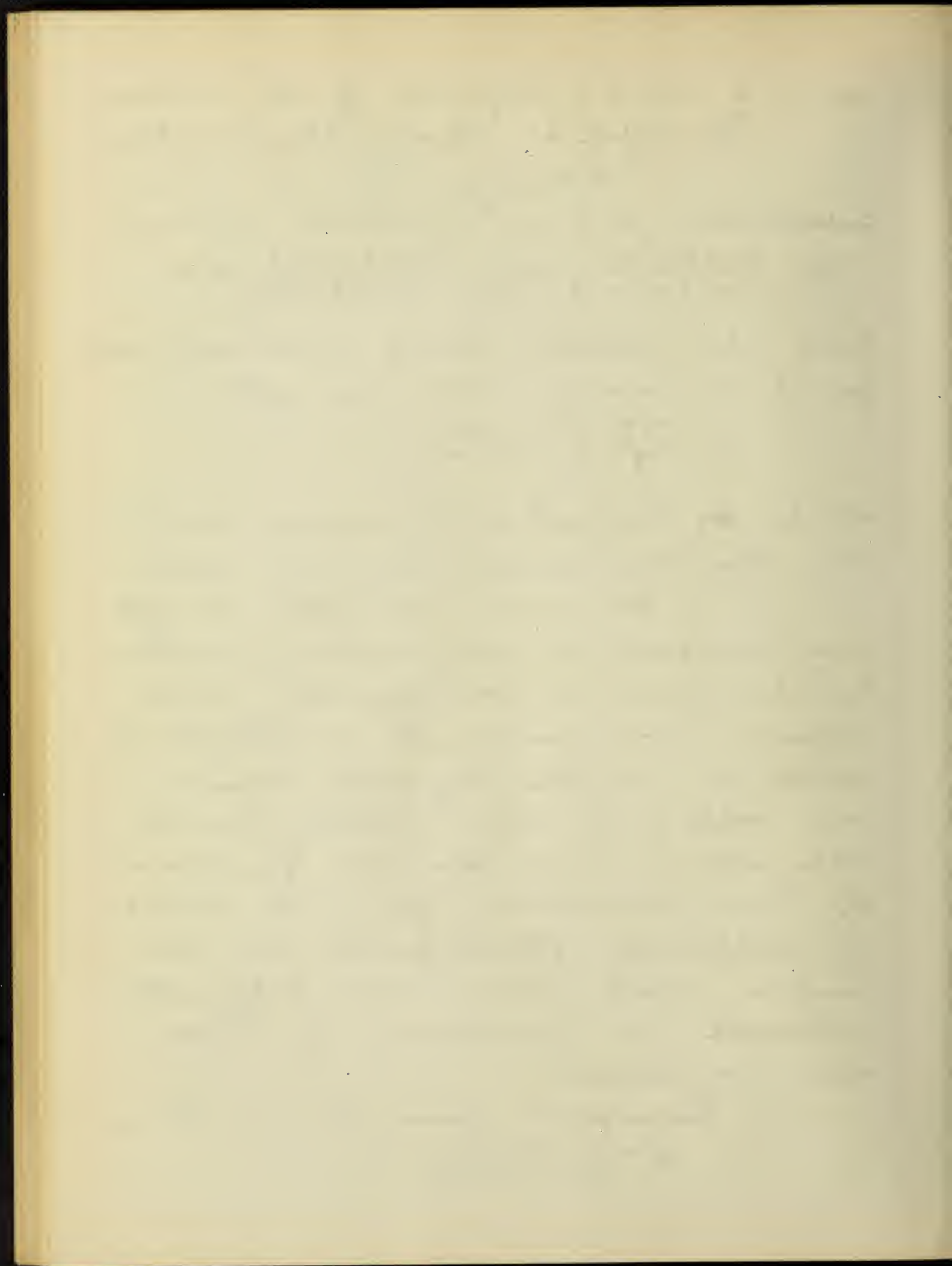
$$\lim_{x \neq 0} \frac{my^4}{m^2y^4 + y^4} = \frac{m}{m^2 + 1}$$

which by Proposition I shows that the double limit does not exist.

4. We were not able to get any method of calculating double limits from du Bois-Reymond whose article we mentioned in Chapter I section 10. du Bois-Reymond missed our idea of the double limit altogether and studied functions of two variables from an entirely different standpoint. An example will show our different interests in functions of two real variables.

Example 4:- Given the function

$$Z = \frac{x + (x+y)^2}{2x+y - (x+y)^2}$$



If first $x \neq 0$ and then $y \neq 0$, we get the limiting value 0; if first $y \neq 0$ and then $x \neq 0$ we get the value $\frac{1}{2}$. But if we put $y = \lambda x$ and then $x \neq 0$ we get the limiting value $\frac{1}{2+\lambda}$ which has a continuous series of values if λ varies from 0 to ∞ .

In such a case we say the twice taken limits, i.e. $\lim_{x \neq 0} \lim_{y \neq 0} f(x, y)$ and $\lim_{y \neq 0} \lim_{x \neq 0} f(x, y)$ both exist, but the double limit $\lim_{\substack{x \neq 0 \\ y \neq 0}} f(x, y)$ does not exist.

du Bois-Reymond calls such points as the origin in this case "Stetigkeitspunkte" and devotes the paper to discussing functions at such points. He develops a method of constructing all possible values of $f(x, y)$ for $x=0, y=0$, which values he gets in the form of a curve which he calls the "Limitale". He does not discuss functions which have double limits, for such functions have no "Stetigkeitspunkte" with which he is interested.

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5. In conclusion we can only say our general method of calculating double limits rests finally upon our fundamental definition given in Chapter I, section 2. In our work we found the negative test given in first section of this chapter the most useful, for the majority of functions considered had no double limit at the origin. In proving that a certain double limit did exist we have always used the method of section two.

My dear Mr. [illegible]
I have just received your letter of the 10th inst. and am
glad to hear that you are well. I am also well and hope
this letter finds you the same. I have not much news to
write at present. I am still in the same place and
continue to work on my [illegible] and [illegible].
I have not yet had time to write you more fully.
I will do so as soon as I have a chance. I am
very truly yours,
[illegible]

Chapter IV.

Discussion of Special Functions.

In this chapter we expect to discuss a number of special problems. In general the problems used in the preceding chapters for illustrative purposes will not be further discussed here. In a few cases however problems will be repeated here for purposes of more complete discussion than was possible in the preceding pages.

Example 1:- Given the function

$$z = \frac{xy}{x^2 + y^2}.$$

to investigate for a double limit at the origin.

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{xy}{x^2 + y^2} = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{xy}{x^2 + y^2} = 0.$$

$$\lim_{x \rightarrow 0} \frac{x y'}{x^2 + y'^2} = 0, \quad \text{for every constant } y' \neq 0.$$

$$\lim_{y \rightarrow 0} \frac{x' y}{x'^2 + y^2} = 0, \quad \text{" " " " " } x' \neq 0.$$

Thus we see that the twice taken limits

My dear Mr. [Name]
I have the honor to acknowledge the receipt of your letter of the 10th inst. in relation to the [subject] and in reply to inform you that the same has been forwarded to the proper authorities for their consideration. I am, Sir, very respectfully,
Yours truly,
[Signature]





and the single limits for constant x and constant y are all zero, yet the double limit does not exist. For if we put $y = mx$ we have

$$\lim_{x \rightarrow 0} \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}.$$

Since this is a function of m we know by Prop. I that the double limit at the point $x=0, y=0$ does not exist. In Chapter I, sec. 4, we showed that the "sprung" of this function was one, i. e. by approaching along the curve $y = mx$ we get limiting values which vary from $-\frac{1}{2}$ to $+\frac{1}{2}$, as m varies from $-\infty$ to $+\infty$.

If we substitute

$$x = \rho \cos \phi$$

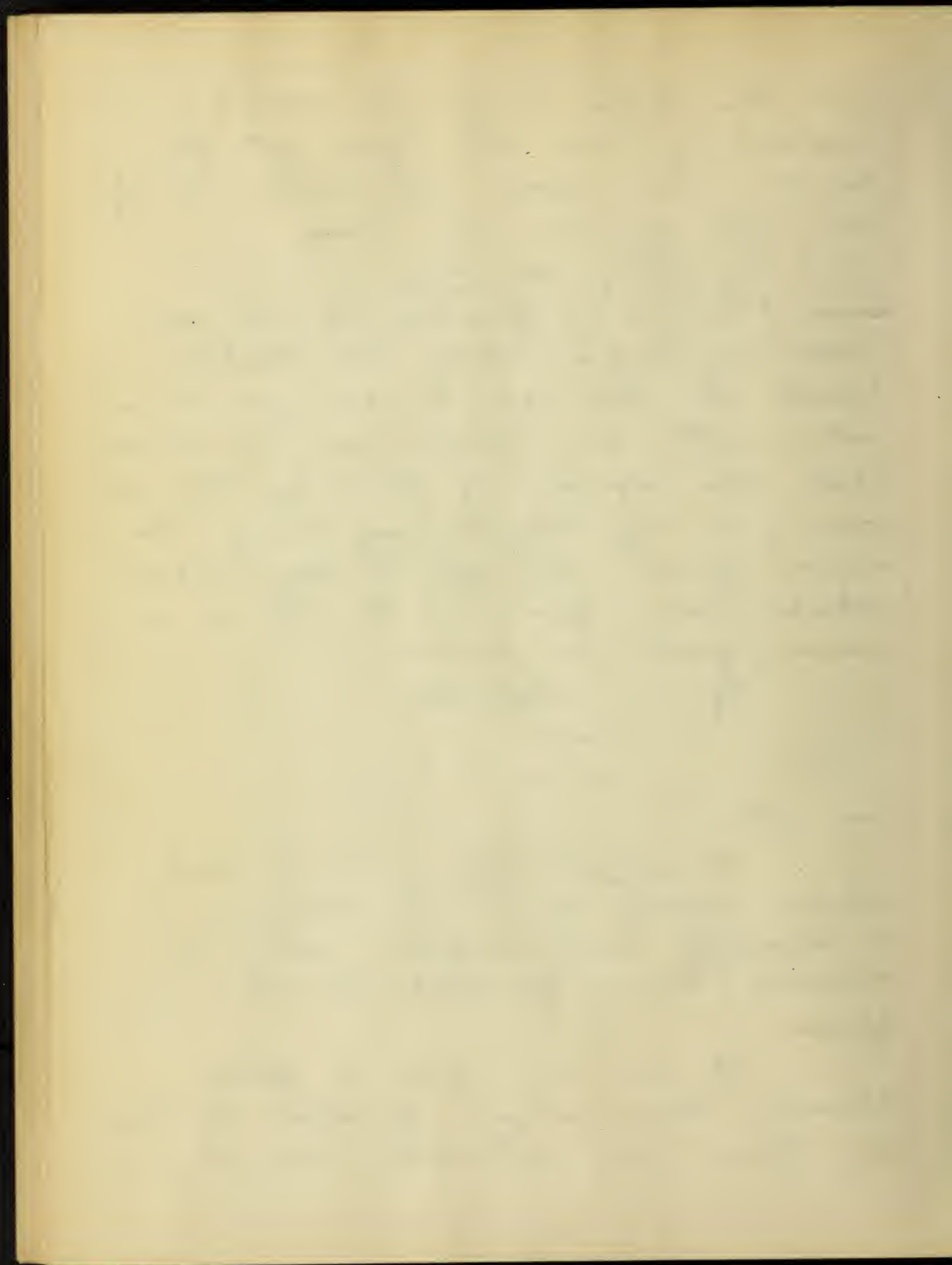
$$y = \rho \sin \phi$$

we get

$$z = \frac{\rho^2 \sin \phi \cdot \cos \phi}{\rho^2 (\cos^2 \phi + \sin^2 \phi)} = \sin \phi \cdot \cos \phi$$

which shows that the surface is a straight line surface, each line element being parallel to the xy -plane.

If we now pass a plane through this surface parallel to the xy -plane and calculate the curves



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of intersection of this plane with the surface, when $y = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, we shall get the four corresponding approximation curves. The data for these curves is as follows:-

$$y = \frac{1}{4}, \quad z = \frac{4x}{16x^2 + 1}$$

z	0	.4	.5	.461	.4	.344	.3	.264	.235
x	0	.125	.25	.375	.5	.625	.75	.875	1.

$$y = \frac{1}{2}, \quad z = \frac{2x}{4x^2 + 1}$$

z	0	.235	.4	.480	.5	.495	.461	.430	.4
x	0	.125	.25	.375	.5	.625	.75	.875	1.

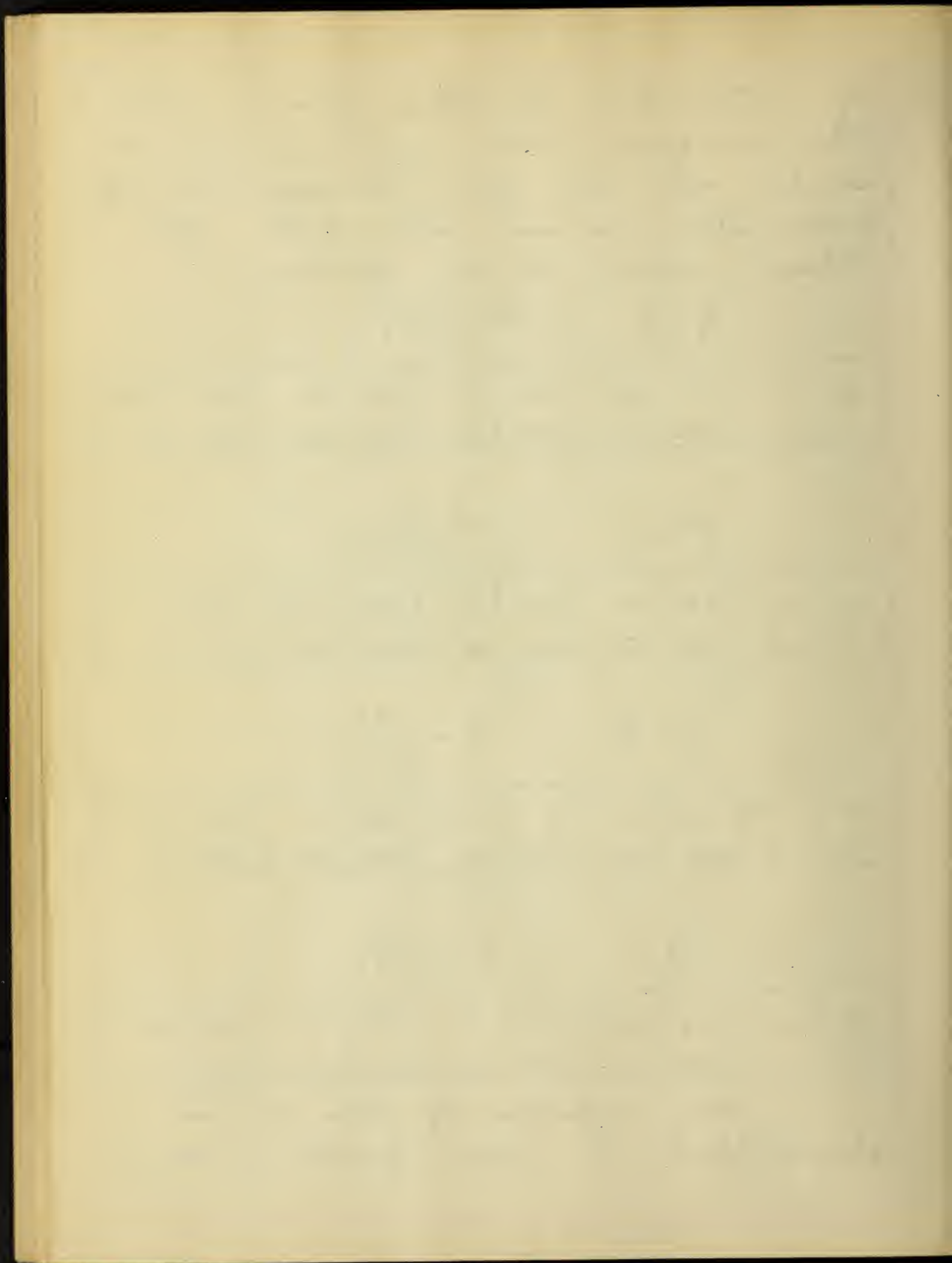
$$y = \frac{3}{4}, \quad z = \frac{12x}{16x^2 + 9}$$

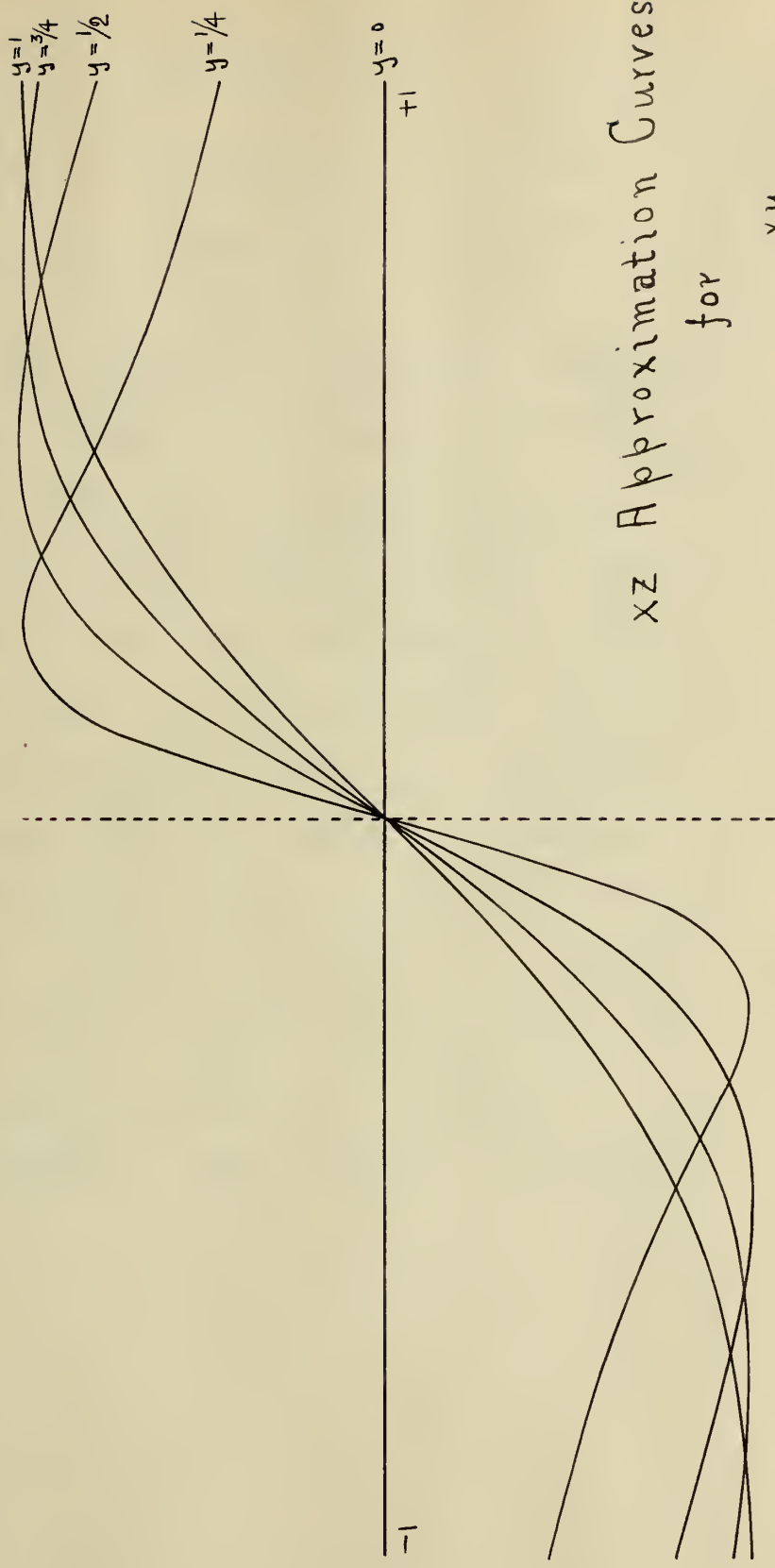
z	0	.166	.3	.4	.461	.490	.5	.494	.48
x	0	.125	.25	.375	.5	.625	.75	.875	1.

$$y = 1, \quad z = \frac{x}{x^2 + 1}$$

z	0	.125	.235	.328	.4	.449	.48	.495	.5
x	0	.125	.25	.375	.5	.625	.75	.875	1.

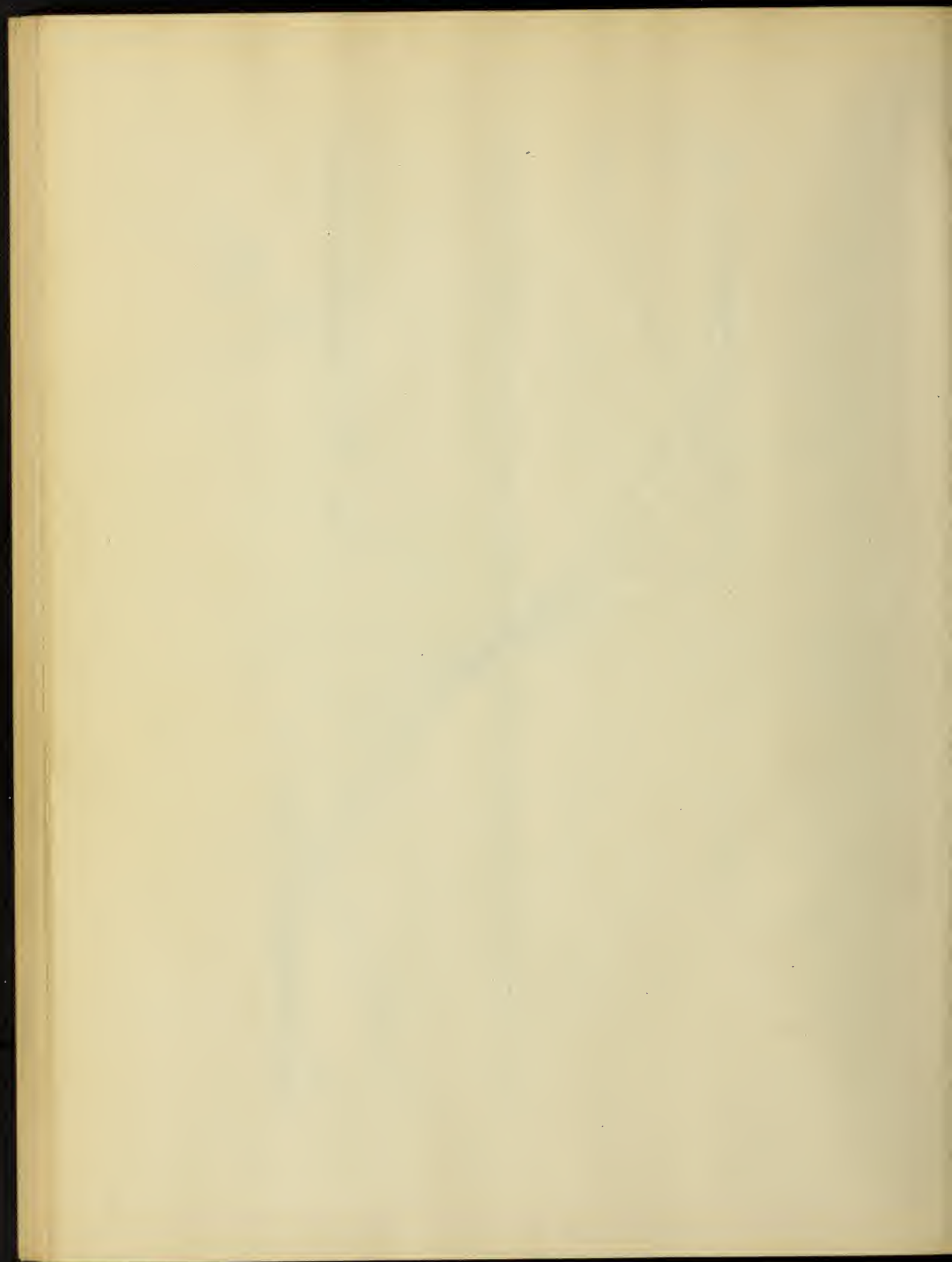
An inspection of the curves shows that at some point z reaches





xz Approximation Curves
for

$$z = \frac{xy}{x^2 + y^2}.$$



the value $+\frac{1}{2}$ and $-\frac{1}{2}$. This maximum and minimum approach the x -axis as y approaches zero. The drawings also show that the function is not uniformly convergent for the interval $-1 < x < +1$.

We constructed a model of this surface, which model is now among the models belonging to the mathematical department of the University of Illinois. Since the surface is a straight line surface, we made a model by stretching silk threads on a frame cut from a brass tube six inches in diameter. However the model does not represent the surface along the z -axis. The following table gives the calculations used in the construction of the model:-

If
then
where

$$y = mx$$

$$z = \frac{m}{1+m^2}$$

$$m = \tan \alpha$$

α	m	z	α	m	z
0°	0	0	112.5	-2.4142	$-.353$
5	$.0875$	$.086$	135	$-1.$	$-1.$
10	$.1763$	$.170$	157.5	$-.4142$	$-.353$
15	$.2680$	$.250$	180	0	0
20	$.3640$	$.321$	202.5	$.4142$	$.353$
25	$.4663$	$.383$	225	$1.$	$1.$
30	$.5774$	$.433$	247.5	2.4142	$.353$
35	$.7002$	$.469$	270	∞	0
40	$.8391$	$.492$	292.5	-2.4142	$-.353$
45	$1.$	$.500$	315	$-1.$	$-1.$
67.5	2.4142	$.353$	337.5	$-.4142$	$-.353$
90	∞	0	360	0	0

Example 2:- Given the function

$$z = \frac{xy^2}{x^2+y^4}$$

to test for double limit at the origin.

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{xy^2}{x^2+y^4} = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{xy^2}{x^2+y^4} = 0$$

$$\lim_{x \rightarrow 0} \frac{xy^2}{x^2+y^4} = 0, \text{ for every constant } y' \neq 0$$

$$\lim_{y \rightarrow 0} \frac{xy^2}{x^2+y^4} = 0, \text{ " " " " } x' \neq 0$$

If we put $y = mx$ then we get

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Jan	100	Jan	100	Jan	100
Feb	100	Feb	100	Feb	100
Mar	100	Mar	100	Mar	100
Apr	100	Apr	100	Apr	100
May	100	May	100	May	100
Jun	100	Jun	100	Jun	100
Jul	100	Jul	100	Jul	100
Aug	100	Aug	100	Aug	100
Sep	100	Sep	100	Sep	100
Oct	100	Oct	100	Oct	100
Nov	100	Nov	100	Nov	100
Dec	100	Dec	100	Dec	100

...





$$\lim_{x \rightarrow 0} \frac{x^3 m^2}{x^2 + m^4 x^4} = \lim_{x \rightarrow 0} \frac{x m^2}{1 + m^4 x^2} = 0.$$

If we put $x = my$ then we get

$$\lim_{y \rightarrow 0} \frac{my^3}{m^2 y^2 + y^4} = \lim_{y \rightarrow 0} \frac{my}{m^2 + y^2} = 0.$$

If we put $y = mx^2$ then we get

$$\lim_{x \rightarrow 0} \frac{m^2 x^5}{x^2 + m^4 x^8} = \lim_{x \rightarrow 0} \frac{m^2 x^3}{1 + m^4 x^6} = 0.$$

If we put $x = my^2$ we have

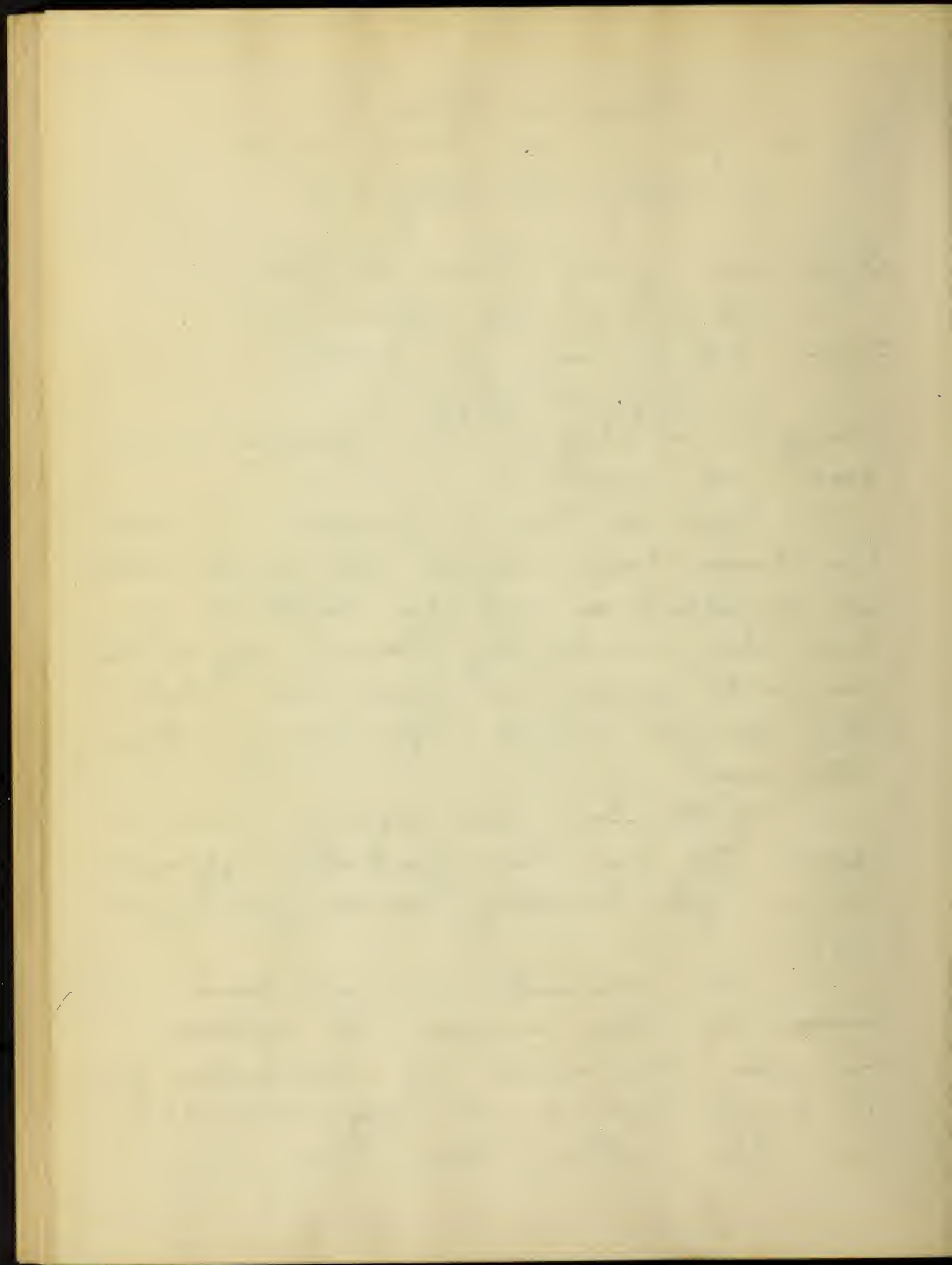
$$\lim_{y \rightarrow 0} \frac{m y^4}{m^2 y^4 + y^4} = \frac{m}{m^2 + 1}.$$

Therefore, by Prop. I, the double limit does not exist.

Here we have a function in which the twice taken limits, the single limits for constant x and for constant y , and the limits by linear approaches are all equal to zero, and still the double limit for $x=0, y=0$ does not exist.

Like Ex. 1 the "spring" here is equal to one. By quadratic approaches we get limiting values from $-\frac{1}{2}$ to $+\frac{1}{2}$.

We constructed a card-board model of this surface. We plotted on cards the curves of intersection for $x = \pm 1, \pm \frac{3}{4}, \pm \frac{2}{4}, \pm \frac{1}{4}, 0$ and for $y = \pm 1, \pm \frac{3}{4}, \pm \frac{1}{2}, \pm \frac{1}{4}, 0$. and after cutting along these curves



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we fitted the cards together so their upper edges represented the surface. The calculations made were as follows:-

$y \backslash x$	-1	-.75	$-.5\frac{1}{4}$	-.50	-.25	$-.06\frac{1}{4}$	0	+.25	+.50	+.75	+1
$\pm .25$	-.062	-.082		-.123	-.235	-.500	0	.235	.123	.082	.062
$\pm .5$	-.235	-.300		-.400	-.500		0	.500	.400	.300	.235
$\pm .75$	-.429	-.480	-.500	-.496	-.371		0	.371	.496	.480	.429
± 1	-.500	-.480		-.400	-.235		0	.235	.400	.480	.500
0	0	0		0	0		0	0	0	0	0

This table gives the value of z for given values of x and y ; thus for $x=1, y=1$, we have $z=.5$. Each curve of intersection parallel to the zx -plane is symmetrical in opposite quadrants. Each curve of intersection parallel to the zy -plane is symmetrical with respect to the x -axis and is either all positive or all negative.

From the data of the above table we made drawings of the approximation curves for $y=\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1. We notice that each curve has a maximum where $z=\frac{1}{2}$ and a minimum where $z=-\frac{1}{2}$. As y becomes small this maximum and

1892
Jan 1st
To balance forward 100.00
By Cash 50.00
100.00

Feb 1st
To Cash 25.00
By Cash 10.00
15.00

Mar 1st
To Cash 30.00
By Cash 15.00
15.00

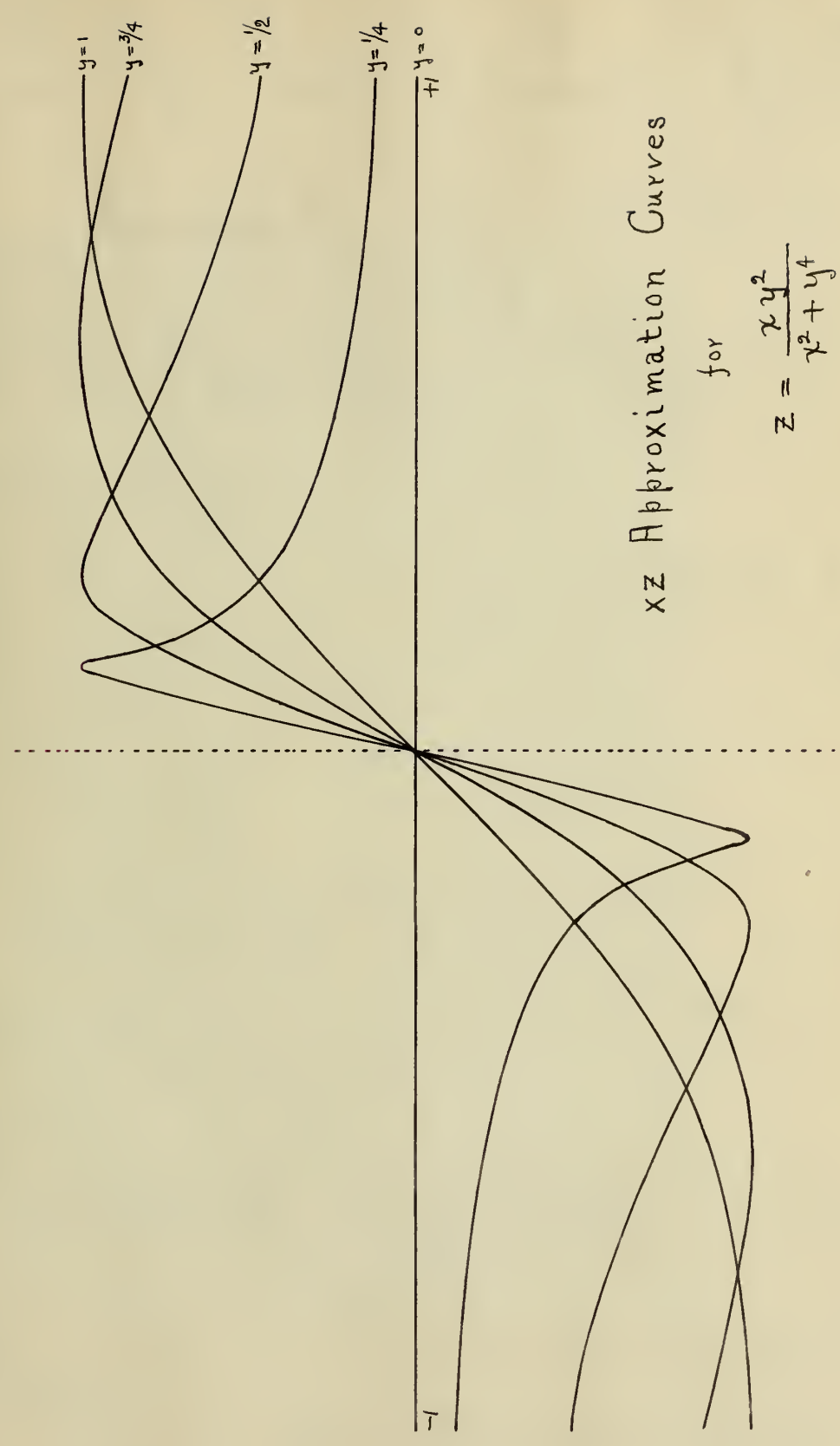
Apr 1st
To Cash 40.00
By Cash 20.00
20.00

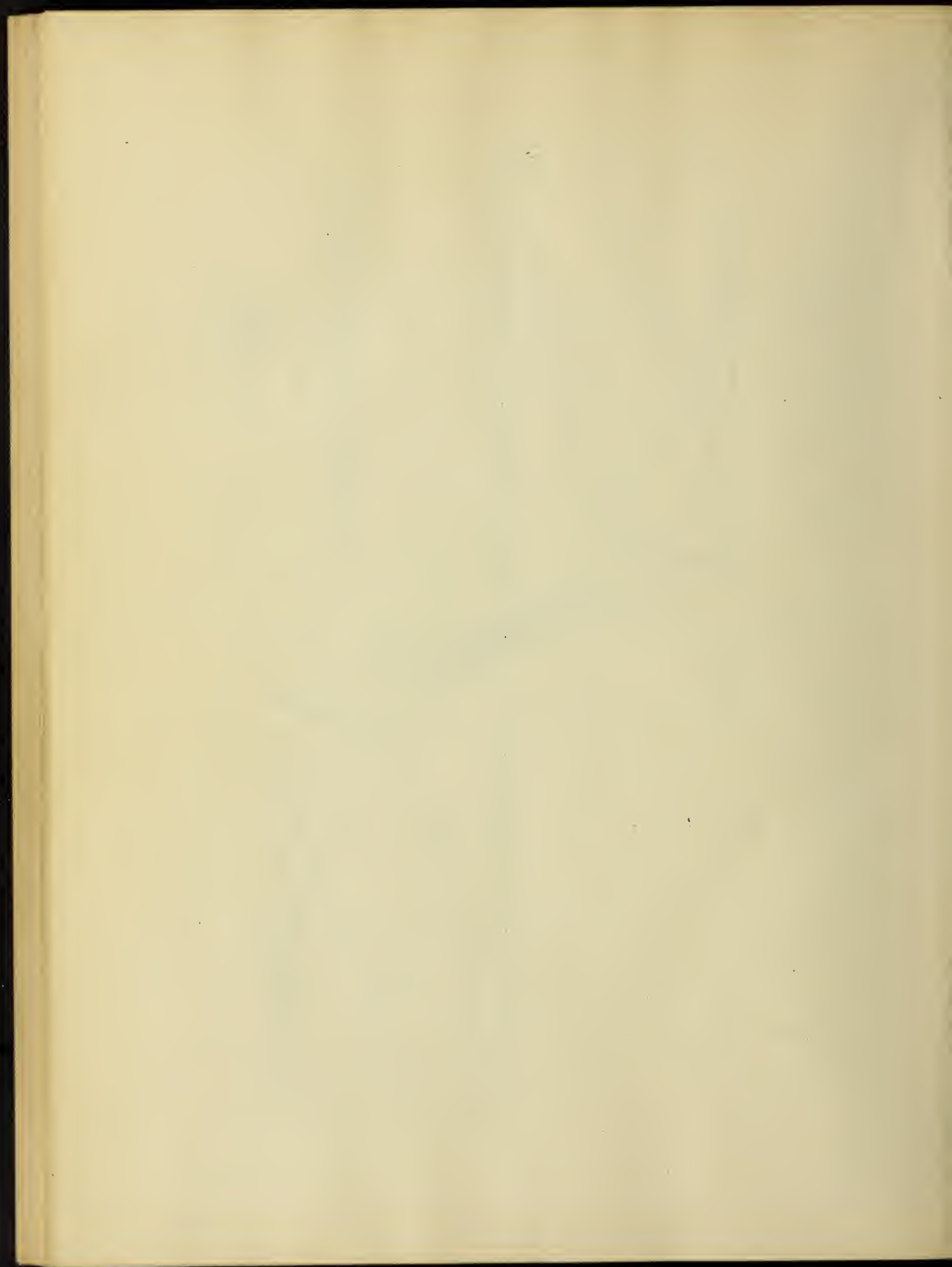
May 1st
To Cash 50.00
By Cash 25.00
25.00

xz Approximation Curves

for

$$z = \frac{xy^2}{x^2 + y^4}$$





minimum approach the axis of x . Therefore the function is not uniformly convergent for the interval $-1 < x < +1$.

Example 3:- Given the function

$$z = \frac{xy}{x+y}$$

to test for the existence of double limit at the origin. Here we have

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{xy}{x+y} = 0.$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{xy}{x+y} = 0.$$

$$\lim_{y \rightarrow 0} f(\bar{x}, y) = 0, \text{ for constant } \bar{x} \neq 0.$$

$$\lim_{x \rightarrow 0} f(x, \bar{y}) = 0, \text{ " " " } \bar{y} \neq 0.$$

If we put

$$y = mx^\mu$$

$$1 \leq \mu < \infty$$

then

$$\lim_{x \rightarrow 0} \frac{mx^{\mu+1}}{x + mx^\mu} = \lim_{x \rightarrow 0} \frac{mx^\mu}{1 + mx^{\mu-1}} = 0.$$

Still the double limit does not exist,

for if we put

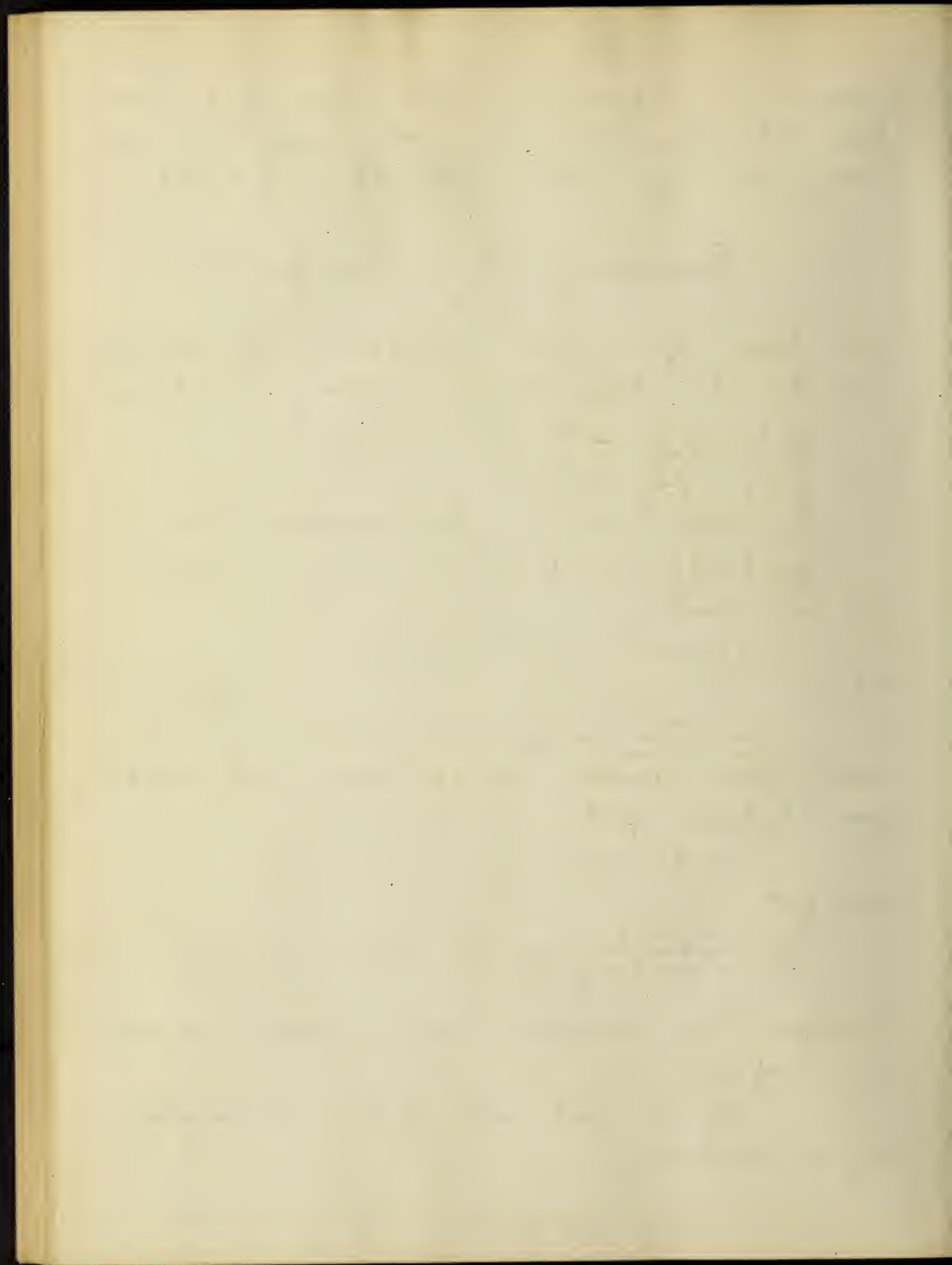
$$x = my^2 - y$$

we get

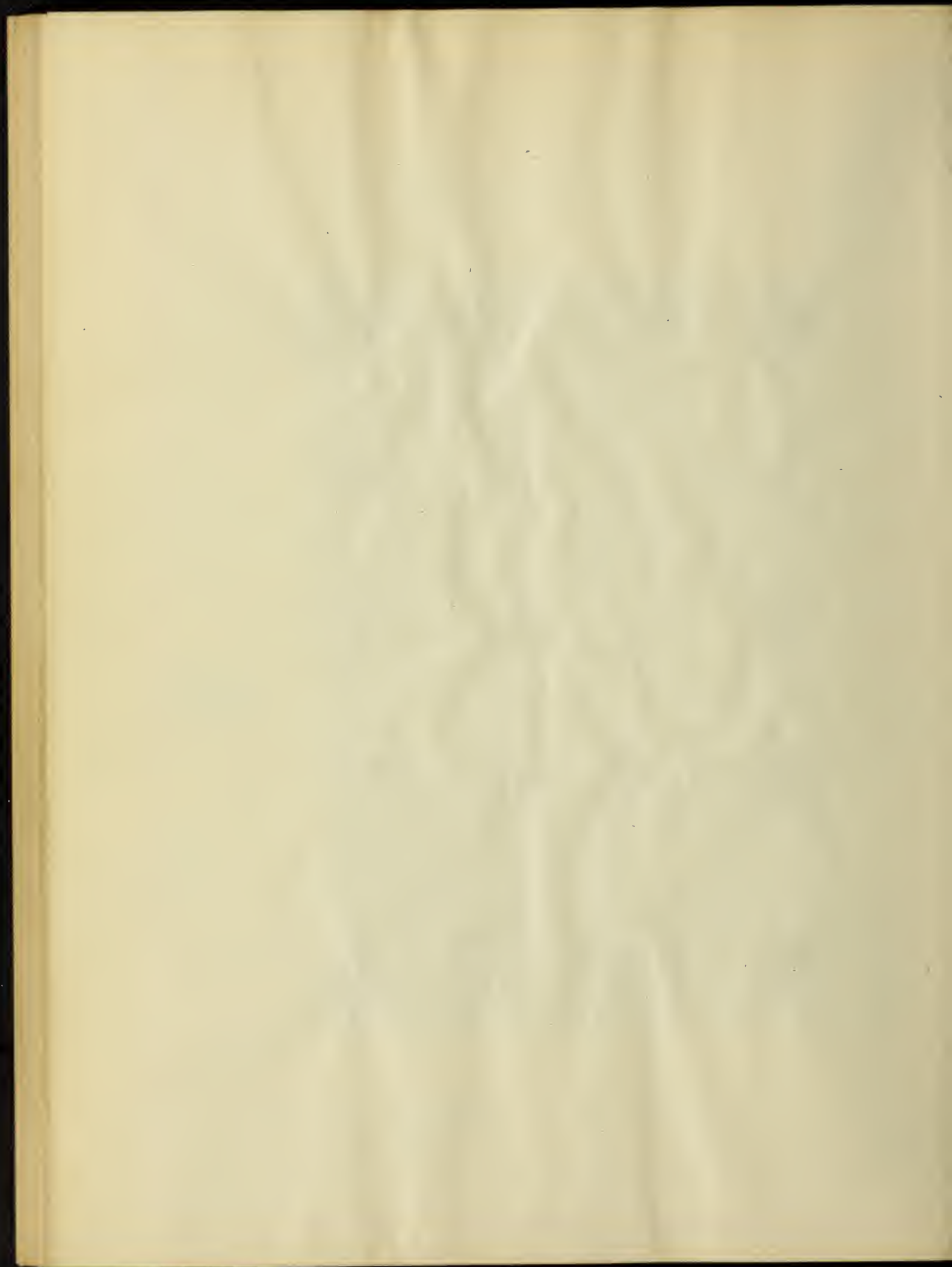
$$\lim_{y \rightarrow 0} \frac{my^3 - y^2}{my^2 - y + y} = \lim_{y \rightarrow 0} \frac{my - 1}{m} = -\frac{1}{m};$$

therefore the double limit does not exist by Prop. I.

If we put into polar coordinates by substituting







$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

then we get

$$z = \frac{\rho^2 \cos \phi \cdot \sin \phi}{\rho(\cos \phi + \sin \phi)} = \rho \frac{\cos \phi \cdot \sin \phi}{\cos \phi + \sin \phi}$$

which shows that the surface is a straight line surface, each straight line element passing through the origin.

We made a model of this surface by stretching silk threads on a frame made of galvanized iron. The frame was made by cutting away all of a circular cylinder, six inches in diameter by seven inches long, except a narrow strip along the intersection of the surface with the cylinder. The following calculations were made in constructing the model.

$$z = \frac{\sin \phi}{1 + \tan \phi}, \text{ for } \rho = 1.$$

ϕ°	315	320	325	330	335	340	345	350	355	360	5
Z	0	-3.99	-1.913	-1.182	-.791	-.537	-.353	-.210	-.095	0	.080
ϕ°	10	15	20	25	30	35	40	45	50	55	60
Z	.147	.204	.250	.287	.316	.337	.349	.353	.349	.337	.316
ϕ°	65	70	75	80	85	90	95	100	105	110	115
Z	.287	.250	.204	.147	.080	0	.080	.147	.204	.250	.287

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Handwritten text in Arabic script, continuing from the previous block. This section also contains mostly illegible text due to the same quality issues. It seems to be a continuation of the same document or a separate entry, with some lines that might be signatures or dates at the bottom.

We plotted four approximation curves from the following calculations:-

$$y = \frac{1}{4} \quad z = \frac{x}{4x+1}$$

x	-8/8	-7/8	-6/8	-5/8	-4/8	-3/8	-2/8	-1/8	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
y	.333	.350	.375	.416	.5	.75	.2	-.25	0	.083	.125	.150	.166	.178	.187	.194	.200

$$y = \frac{2}{4} \quad z = \frac{x}{2x+1}$$

x	-1	-7/8	-6/8	-5/8	-4/8	-3/8	-2/8	-1/8	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
y	1.	1.16	1.5	2.5	∞	-1.33	-.5	-.66	0	.1	.166	.214	.25	.277	.3	.318	.333

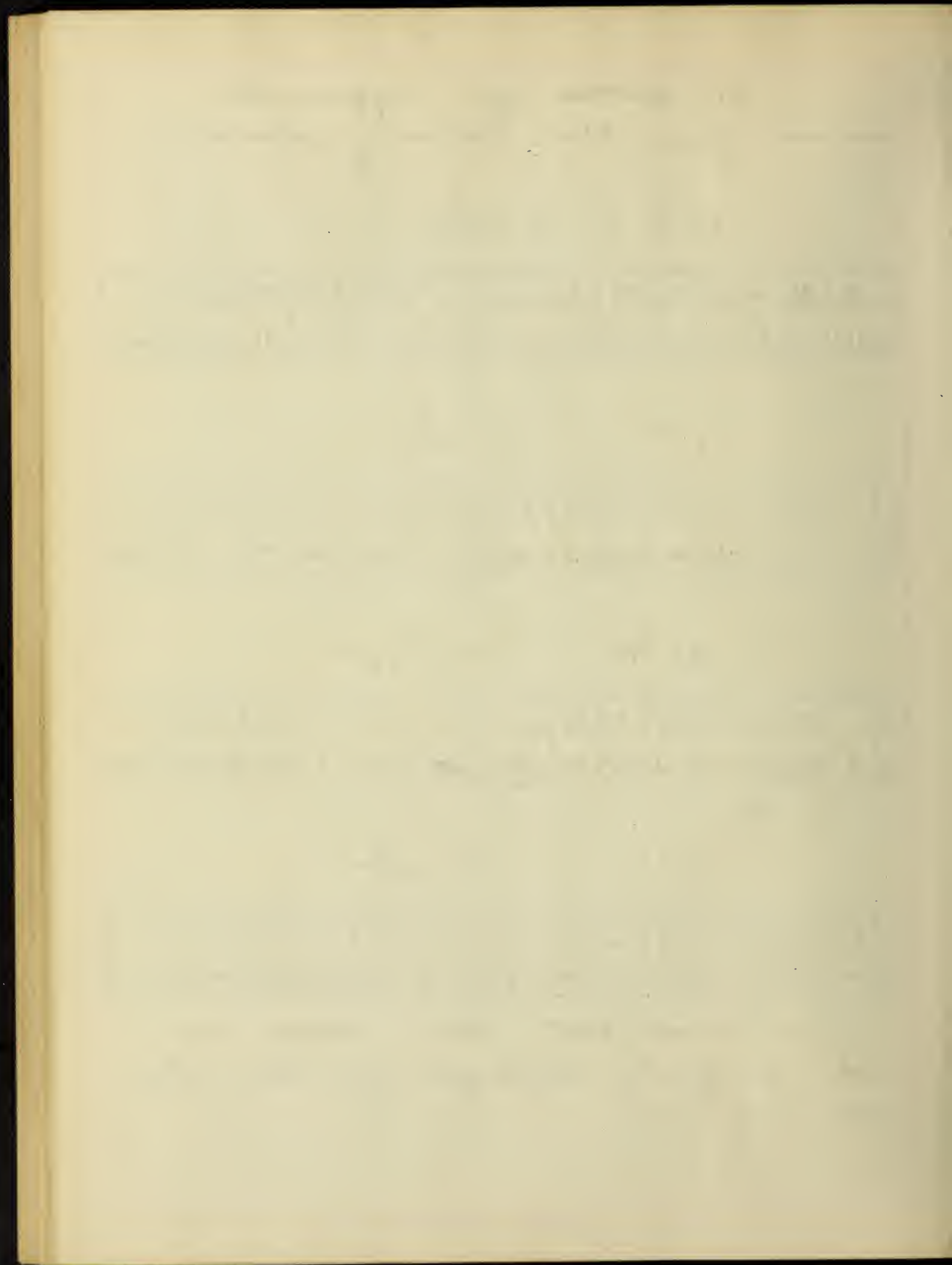
$$y = \frac{3}{4} \quad z = \frac{3x}{4x+3}$$

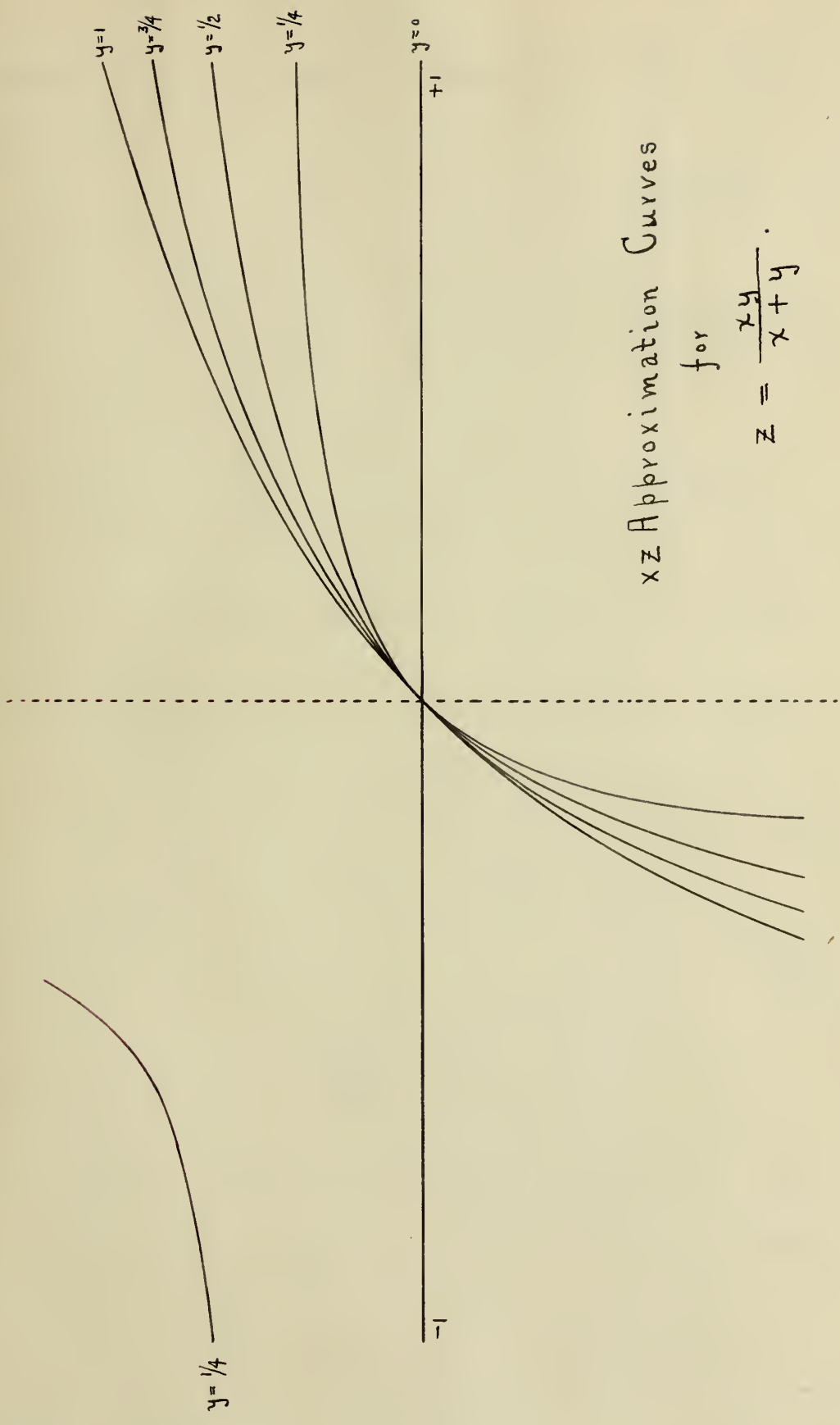
x	-1	-7/8	-6/8	-5/8	-4/8	-3/8	-2/8	-1/8	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
y	3	5.25	∞	-3.75	-1.5	-.75	-.375	-.150	0	.107	.187	.25	.3	.340	.375	.403	.428

$$y = 1 \quad z = \frac{x}{x+1}$$

x	-8/8	-7/8	-6/8	-5/8	-4/8	-3/8	-2/8	-1/8	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
y	∞	-7	-3	-1.66	-1.	-.6	-.333	-.142	0	.111	.2	.272	.333	.384	.428	.466	.5

We see that these curves are not uniformly convergent for the interval $-1 < x < +1$.





xz Approximation Curves

for
$$z = \frac{xy}{x+y}.$$

The final result is proved only for curves
of this type $\mathcal{K} = m y^2$ or $y = m x^2$
logarithmic $\mathcal{K} = a_0 y^m + a_1 y^{m-1} + \dots + a_m$

Example 4:- Given the function

$$Z = \frac{x y^n}{x^2 + y^{2n}}$$

to test for the existence of the double limit at the origin.

Examples 1 and 2 are special cases of this general form.

This function is continuous at the origin with respect to x alone and y alone; for

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x y^n}{x^2 + y^{2n}} = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x y^n}{x^2 + y^{2n}} = 0.$$

Let $y = m x^a$ and then we have

$$\lim_{x \rightarrow 0} \frac{x m^a x^{an}}{x^2 + m^{2n} x^{2an}} = 0 \quad ; \quad \text{where } a \leq n.$$

For $x = m y^b$, we have

$$\lim_{y \rightarrow 0} \frac{m y^{n+b}}{m^2 y^{2b} + y^{2n}} = 0 \quad ; \quad \text{where } b < n,$$

$$= \frac{m}{m^2 + 1} \quad ; \quad \text{where } b = n.$$

Here we have a function where, by every approach to the origin along a continuous curve of less than the n^{th} degree, we always have the same limiting value 0 and still the double limit does not exist; for along the curve $x = m y^n$, we do not come to the limiting value 0.

Example 5:- Given the function

$$Z = \frac{xy}{x^2 + y^3}$$

to test for the existence of the double limit at the origin.

This function is continuous with respect to each variable separately; for

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{xy}{x^2 + y^3} = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{xy}{x^2 + y^3} = 0.$$

Let $y = mx$ and we have

$$\lim_{x \rightarrow 0} \frac{mx^2}{x^2 + m^3x^3} = \lim_{x \rightarrow 0} \frac{m}{1 + m^3x} = m.$$

Therefore the double limit does not exist, by Prop. I.

Since by linear approaches to the origin we may get limiting values which vary from $-\infty$ to $+\infty$, we see at once that this function has an infinite sprung at the point $x=0, y=0$.

Example 6:- Given the function

$$Z = \frac{xy}{x^3 + y^3}$$

to test for the existence of the double limit at the origin.

This function also has an infinite sprung, but differs from the last example in that by continuous linear approaches we can

My dear mother
I have just received your letter
and am very glad to hear from you.
I am well and hope this finds you the same.

I have been thinking of you
very much lately and wondering
how you are getting on.
I hope you are happy and
content. I have been very busy
lately but I always find time
to think of my friends.
I hope to hear from you soon.
I am your affectionate son,
John Doe

I have been thinking of you
very much lately and wondering
how you are getting on.
I hope you are happy and
content. I have been very busy
lately but I always find time
to think of my friends.
I hope to hear from you soon.
I am your affectionate son,
John Doe

obtain as limiting values only $-\infty, 0, +\infty$.
We have here

$$\lim_{x \neq 0} \lim_{y \neq 0} \frac{xy}{x^3 + y^3} = \lim_{y \neq 0} \lim_{x \neq 0} \frac{xy}{x^3 + y^3} = 0.$$

$$\lim_{x \neq 0} \frac{xy}{x^3 + y^3} = 0; \text{ for every constant } y \neq 0.$$

$$\lim_{y \neq 0} \frac{xy}{x^3 + y^3} = 0; \text{ " " " " } x \neq 0.$$

Let $y = mx$ and then we have

$$\lim_{x \neq 0} \frac{mx^2}{x^3 + m^3x^3} = \lim_{x \neq 0} \frac{m}{x(1+m^3)} = \infty; m \neq 0 \text{ or } \infty.$$

$$= 0; m = 0 \text{ or } \infty.$$

Therefore the double limit does not exist for by definition it must always be a definite finite number.

Let $x = \rho \cos \phi$ and $y = \rho \sin \phi$ and we get

$$\lim_{\rho \neq 0} \frac{\rho^2(\cos \phi \cdot \sin \phi)}{\rho^3(\cos^3 \phi + \sin^3 \phi)} = \infty,$$

except for $\phi = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ$. Thus by linear approaches to the origin we get as limiting values only 0 and $\pm\infty$.
More specifically we get

$$\lim_{\rho \neq 0} \frac{\cos \phi \cdot \sin \phi}{\rho(\cos^3 \phi + \sin^3 \phi)} = 0; \phi = 0, 90^\circ, 180^\circ \text{ or } 270^\circ$$

$$= +\infty; 0 < \phi < 90^\circ$$

$$= -\infty; 90^\circ < \phi < 135^\circ$$

$$= +\infty; 135^\circ < \phi < 180^\circ$$

$$= -\infty; 180^\circ < \phi < 270^\circ$$

$$= +\infty; 270^\circ < \phi < 315^\circ$$

$$= -\infty; 315^\circ < \phi < 360^\circ$$

Example 7:- Given the function

$$z = \frac{ax + by}{x + y}$$

to test for the existence of the double limit at the origin.

Here we have a function which has a different limit at the origin when regarded as a function of x alone than when regarded as a function of y alone.

$$\lim_{x \rightarrow 0} \frac{ax + by}{x + y} = b; \text{ for constant } y \neq 0.$$

$$\lim_{y \rightarrow 0} \frac{ax + by}{x + y} = a; \quad " \quad " \quad x \neq 0.$$

Let $y = mx$ and then we have

$$\lim_{x \rightarrow 0} \frac{ax + bmx}{x + mx} = \frac{a + bm}{1 + m}.$$

Therefore the double limit does not exist by Prop. I.

Clearing of fractions, we get

$$by + ax - yz - xz = 0$$

which is the equation of an hyperbola for either x or y regarded as constant. Therefore the approximation curves parallel to either the xz -plane or the yz -plane are hyperboles.

Let $x = \rho \cos \phi$ and $y = \rho \sin \phi$, and then we have

$$z = \frac{a \cos \phi + b \sin \phi}{\cos \phi + \sin \phi}$$

which shows this to be a straight

line surface the elements being parallel to the xy -plane.

For $\phi = 0$ or 180 , $z = a$

" $\phi = 90$ " 270 , $z = b$

" $\phi = 45$, $z = \frac{1}{2}(a+b)$

" $\phi = \tan^{-1}(-\frac{a}{b})$, $z = 0$

Example 8:- Given the function

$$z = \frac{x + (x+y)^2}{2x + y - (x+y)^2}$$

to test for the existence of the double limit at the origin.

Here we meet for the first time a function of x, y which, regarded as a function of either alone, gives us in the limit a function of the other variable, the limiting point being as before the origin. Here we have

$$\lim_{x \neq 0} \lim_{y \neq 0} \frac{x + (x+y)^2}{2x + y - (x+y)^2} = \lim_{x \neq 0} \frac{1+x}{2-x} = \frac{1}{2}.$$

$$\lim_{y \neq 0} \lim_{x \neq 0} \frac{x + (x+y)^2}{2x + y - (x+y)^2} = \lim_{y \neq 0} \frac{y}{1-y} = 0.$$

Therefore the double limit does not exist by Prop. I.

$$\lim_{x \neq 0} \frac{x + (x+\bar{y})^2}{2x + \bar{y} - (x+\bar{y})^2} = \frac{\bar{y}}{1-\bar{y}}, \text{ for constant } y \neq 0.$$

$$\lim_{y \neq 0} \frac{\bar{x} + (\bar{x}+y)^2}{2\bar{x} + y - (\bar{x}+y)^2} = \frac{1+\bar{x}}{2-\bar{x}}, \text{ " " } x \neq 0.$$

If we let $y = mx$, then we have

1871
The first of the year was a very
cold one, and the weather was
very disagreeable. The snow
was very deep, and the wind
was very strong.

The second of the year was a
very warm one, and the weather
was very pleasant. The snow
was very deep, and the wind
was very strong. The third of
the year was a very cold one,
and the weather was very
disagreeable. The snow was
very deep, and the wind was
very strong.

The fourth of the year was a
very warm one, and the weather
was very pleasant. The snow
was very deep, and the wind
was very strong. The fifth of
the year was a very cold one,
and the weather was very
disagreeable. The snow was
very deep, and the wind was
very strong.

$$\lim_{x \neq 0} \frac{x + (x + mx)^2}{2x + m - (x + mx)^2} = \lim_{x \neq 0} \frac{1 + x(1+m)^2}{2 + m - x(1+m)^2} = \frac{1}{2+m}.$$

If we substitute $x = \rho \cos \phi$, $y = \rho \sin \phi$, then

$$\begin{aligned} \lim_{\rho \neq 0} \frac{\rho \cos \phi + \rho^2 (\cos \phi + \sin \phi)^2}{2\rho \cos \phi + \rho \sin \phi - \rho^2 (\cos \phi + \sin \phi)^2} \\ = \lim_{\rho \neq 0} \frac{\cos \phi + 2\rho \cos \phi \cdot \sin \phi}{2 \cos \phi + \sin \phi - 2\rho \cos \phi \cdot \sin \phi} \\ = \lim_{\rho \neq 0} \frac{1 - 2\rho \sin \phi}{2 + \tan \phi - 2\rho \sin \phi} = \frac{1}{2 + \tan \phi} \end{aligned}$$

as we might expect from above where $m = \tan \phi$.

The equation of intersection of this surface with a plane through the z -axis is

$$2z + z \cdot \tan \phi - 2\rho \cdot z \sin \phi = 1 + 2\rho \sin \phi,$$

which is a quadratic in ρ and z , and is an hyperbola since $(H^2 - AB) > 0$, where $\phi \neq 0$ or 180° .

By clearing of fractions in the original form we get

$$2zx + zy - z(x+y)^2 = x + (x+y)^2.$$

For constant y this is a cubic in z and x ; for constant x , a cubic in z and y . Thus the plane intersections parallel to both zx -plane and zy -plane are cubic curves.

1871

London and
1871

1871

1871

1871

1871

Example 9:- Given the function

$$Z = \frac{y^5 + xy^3}{x^2}$$

to test for the existence of the double limit at the origin.

Here we have a function in which we obtain the limiting value zero by every linear approach except along the y-axis, in which case we get ∞ . Here we have

$$\lim_{x \neq 0} \lim_{y \neq 0} \frac{y^5 + xy^3}{x^2} = 0,$$

$$\lim_{y \neq 0} \lim_{x \neq 0} \frac{y^5 + xy^3}{x^2} = \infty,$$

$$\lim_{x \neq 0} \frac{y^5 + xy^3}{x^2} = \infty, \text{ for constant } y \neq 0,$$

$$\lim_{y \neq 0} \frac{y^5 + xy^3}{x^2} = 0, \quad " \quad " \quad x \neq 0.$$

Let $y = mx$ and then we have

$$\lim_{x \neq 0} \frac{m^5 x^5 + x m^3 x^3}{x^2} = \lim_{x \neq 0} (m^5 x^3 + m^3 x) = 0.$$

Let $y^5 = mx^2$ and then we have

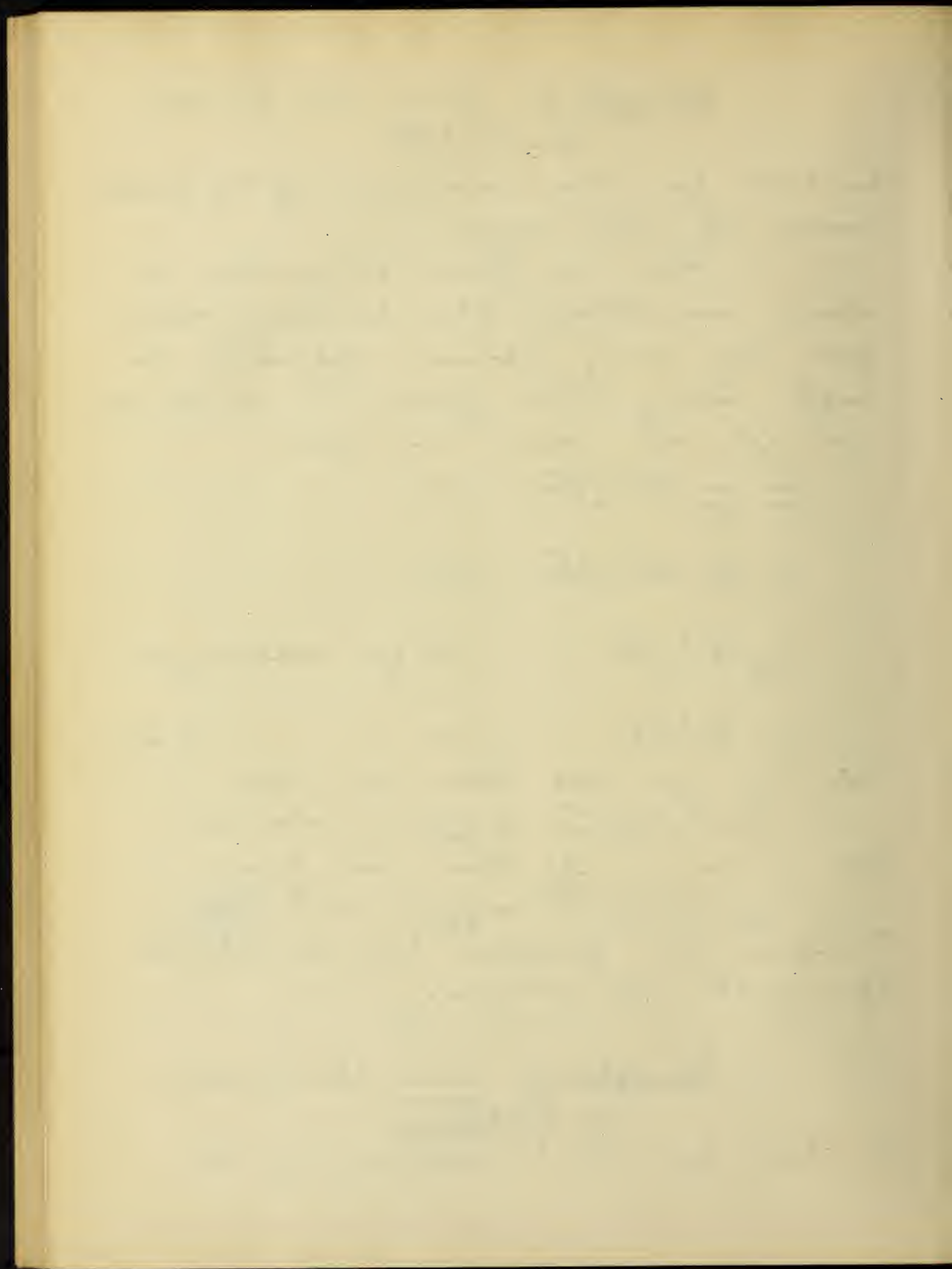
$$\lim_{x \neq 0} \frac{mx^2 + x(m^{\frac{2}{5}} x^{\frac{1}{5}})^3}{x^2} = \lim_{x \neq 0} (m + m^{\frac{3}{5}} x^{\frac{1}{5}}) = m.$$

Therefore this function has an infinite spring at the origin.

Example 10:- Given the function

$$Z = \frac{x^4 + y\sqrt{y^3 - x^3}}{y^4 + x\sqrt{x^5 + y^5}}$$

to test for the existence of the



double limit at the origin.

Here, as in Ex. 8, the single limits when x is regarded as a constant or when y is regarded as a constant depend upon the constant selected for these variables. We have here

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^4 + y \sqrt{y^3 - x^3}}{y^4 + x \sqrt{x^5 + y^5}} = \lim_{x \rightarrow 0} \frac{x^4}{x^3 \sqrt{x}} = \lim_{x \rightarrow 0} \frac{x^4}{x^{\frac{7}{2}}} = \lim_{x \rightarrow 0} \sqrt{x} = 0$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \frac{y \sqrt{y^3}}{y^4} = \lim_{y \rightarrow 0} \frac{y^{\frac{5}{2}}}{y^4} = \lim_{y \rightarrow 0} \frac{1}{y^{\frac{3}{2}}} = \infty$$

$$\lim_{y \rightarrow 0} \frac{x^4 + y \sqrt{y^3 - x^3}}{y^4 + x \sqrt{x^5 + y^5}} = \sqrt{x}, \text{ for constant } x \neq 0.$$

$$\lim_{x \rightarrow 0} \frac{x^4 + y \sqrt{y^3 - x^3}}{y^4 + x \sqrt{x^5 + y^5}} = y^{-\frac{3}{2}}, \text{ " " } y \neq 0.$$

Let $y = mx$ and we have

$$\lim_{x \rightarrow 0} \frac{x^4 + mx \sqrt{m^3 x^3 - x^3}}{m^4 x^4 + x \sqrt{m^5 x^5 + x^5}} = \lim_{x \rightarrow 0} \frac{x^2 + m \sqrt{m^3 x - x}}{m^4 x^2 + x \sqrt{m^5 x + x}} = 0.$$

Let $y = x + mx^3$ and we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^4 + (x + mx^3) \sqrt{(x + mx^3)^3 - x^3}}{(x + mx^3)^4 + x \sqrt{x^5 + (x + mx^3)^5}} &= \\ &= \lim_{x \rightarrow 0} \frac{x^{\frac{1}{2}} + (1 + mx) \sqrt{3m + 3m^2 x^2 + m^3 x^4}}{x^{\frac{1}{2}} (1 + mx^3)^4 + \sqrt{1 + (1 + mx^2)^5}} \\ &= \sqrt{\frac{3m}{2}}. \end{aligned}$$

Therefore the double limit does not exist by Prop. I. Here we have a function in which the double limit at the

1. The first part of the paper is devoted to a general
discussion of the problem. It is shown that the
problem is of great importance and that it has
not been completely solved. The author then
presents a new method for solving the problem.

2. In the second part of the paper, the author
applies the new method to a specific case. It is
shown that the method is very effective and that
it can be used to solve a wide range of problems.

3. The third part of the paper is devoted to a
discussion of the results. It is shown that the
new method is very effective and that it can be
used to solve a wide range of problems. The
author then presents a new method for solving the
problem.

4. In the fourth part of the paper, the author
applies the new method to a specific case. It is
shown that the method is very effective and that
it can be used to solve a wide range of problems.

origin does not exist and where the spring is infinite. Approaching the origin along the x -axis we obtain the limit zero, while approaching along the y -axis we get ∞ . By approaches along the curve $y = x + mx^3$ we get limits all the way from 0 to ∞ .

Example 11:- Given the function

$$Z = \frac{ax^2 + bxy + cy^2}{a_1x^2 + b_1xy + c_1y^2}$$

to test for double limit at the origin.

This function has some interesting special cases. Here we have

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \frac{a}{a_1},$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \frac{c}{c_1},$$

$$\lim_{x \rightarrow 0} f(x, \bar{y}) = \frac{c}{c_1}, \text{ for constant } y \neq 0.$$

$$\lim_{y \rightarrow 0} f(\bar{x}, y) = \frac{a}{a_1}, \text{ " " } x \neq 0.$$

Let $y = mx$ and we have

$$\lim_{x \rightarrow 0} \frac{ax^2 + mbx^2 + cm^2x^2}{a_1x^2 + mb_1x^2 + c_1m^2x^2} = \frac{a + mb + m^2c}{a_1 + mb_1 + m^2c_1}$$

Therefore the double limit does not exist by Prop. I.

If $\frac{a}{a_1} = \frac{c}{c_1} \neq \frac{b}{b_1}$, then the twice taken limits, and limits for constant x and constant y are all equal. For

example take

$$Z = \frac{4x^2 + 5xy + 6y^2}{2x^2 + xy + 3y^2}.$$

Then

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} f(x, \bar{y}) = \lim_{y \rightarrow 0} f(\bar{x}, y) = 2$$

and still the double limit at the origin does not exist.

Example 12:- Given the function

$$Z = \frac{ay^2 + bx + cy}{a_1y^2 + b_1x + c_1y}$$

to test for the existence of the double limit at the origin.

Here we have another function interesting for some of its special cases. By certain restrictions upon the coefficients, we can get a special case where by the twice taken limits and by approaches along curves $y = mx^n$ or $x = my^n$, $0 < n < \infty$, we always get the same limit, yet the double limit does not exist. We have here

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{ay^2 + bx + cy}{a_1y^2 + b_1x + c_1y} = \lim_{x \rightarrow 0} \frac{bx}{b_1x} = \frac{b}{b_1},$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{ay^2 + bx + cy}{a_1y^2 + b_1x + c_1y} = \lim_{y \rightarrow 0} \frac{ay^2 + cy}{a_1y^2 + c_1y} = \frac{c}{c_1},$$

$$\lim_{x \rightarrow 0} f(x, \bar{y}) = \frac{a\bar{y} + c}{a_1\bar{y} + c_1}, \text{ for constant } y \neq 0,$$

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$$\lim_{y \neq 0} f(\bar{x}, y) = \frac{b\bar{x}}{b_1\bar{x}} = \frac{b}{b_1}, \text{ for constant } \bar{x} \neq 0.$$

If $p = \frac{b}{b_1} = \frac{c}{c_1} \neq \frac{a}{a_1}$ and $b=c, b_1=c_1$, then if we let $y = mx^n$, where $0 < n < \infty$ we get

$$\lim_{x \neq 0} \frac{a m^2 x^{2n} + bx + c m x^n}{a_1 m^2 x^{2n} + b_1 x + c_1 m x^n} = \frac{b}{b_1} = p.$$

If we let $x = my^m$ then we have

$$\lim_{y \neq 0} \frac{a y^2 + b m y^m + c y}{a_1 y^2 + b_1 m y^m + c_1 y} = \frac{c}{c_1} = p.$$

Thus under these restrictions, we have

$$\begin{aligned} \lim_{x \neq 0} \lim_{y \neq 0} f(x, y) &= \lim_{y \neq 0} \lim_{x \neq 0} f(x, y) = \lim_{y \neq 0} f(\bar{x}, y) = \lim_{x \neq 0} f(x, m x^n) \\ &= \lim_{y \neq 0} f(m y^m, y) = p, \end{aligned}$$

and still the double limit does not exist, for if we substitute

$y = mx^2 - x$, we have

$$\lim_{x \neq 0} \frac{a(m x^4 - 2 m x^3 + x^2) + bx + c(m x^2 - x)}{a_1(m x^4 - 2 m x^3 + x^2) + b_1 x + c_1(m x^2 - x)}$$

$$= \lim_{x \neq 0} \frac{a(m x^2 - 2 m x + 1) + c m}{a_1(m x^2 - 2 m x + 1) + c_1 m} = \frac{a + c m}{a_1 + c_1 m}$$

since $b=c$ and $b_1=c_1$.

Example 13:- Given the function

$$z = \frac{5y^2 + 4x + 4y}{3y^2 + 2x + 2y}$$

to test for the existence of the double limit at the origin.

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My dear Sir,
I have the honor to acknowledge the receipt of your letter of the 10th inst. in relation to the matter of the
and in answer to inform you that the same has been forwarded to the proper authorities for their consideration.
I am, Sir, very respectfully,
Your obedient servant,
J. H. [Signature]

69.

This is simply a special case of Ex. 12. We have here

$$\lim_{x \neq 0} \lim_{y \neq 0} \frac{5y^2 + 4x + 4y}{3y^2 + 2x + 2y} = 2,$$

$$\lim_{y \neq 0} \lim_{x \neq 0} \frac{5y^2 + 4x + 4y}{3y^2 + 2x + 2y} = 2,$$

$$\lim_{x \neq 0} f(x, \bar{y}) = \frac{5\bar{y} + 4}{3\bar{y} + 2}, \text{ constant } y \neq 0,$$

$$\lim_{y \neq 0} f(\bar{x}, y) = \frac{4\bar{x}}{2\bar{x}} = 2, \text{ constant } x \neq 0,$$

If we let $y = mx$, then we have

$$\lim_{x \neq 0} \frac{5m^2x^2 + 4x + 4mx}{3m^2x^2 + 2x + 2mx} = 2.$$

If we let $x = my$, then we have

$$\lim_{y \neq 0} \frac{5y^2 + 4my + 4y}{3y^2 + 2my + 2y} = 2.$$

If we let $y = mx^n$ where $1 < n < \infty$, then we have

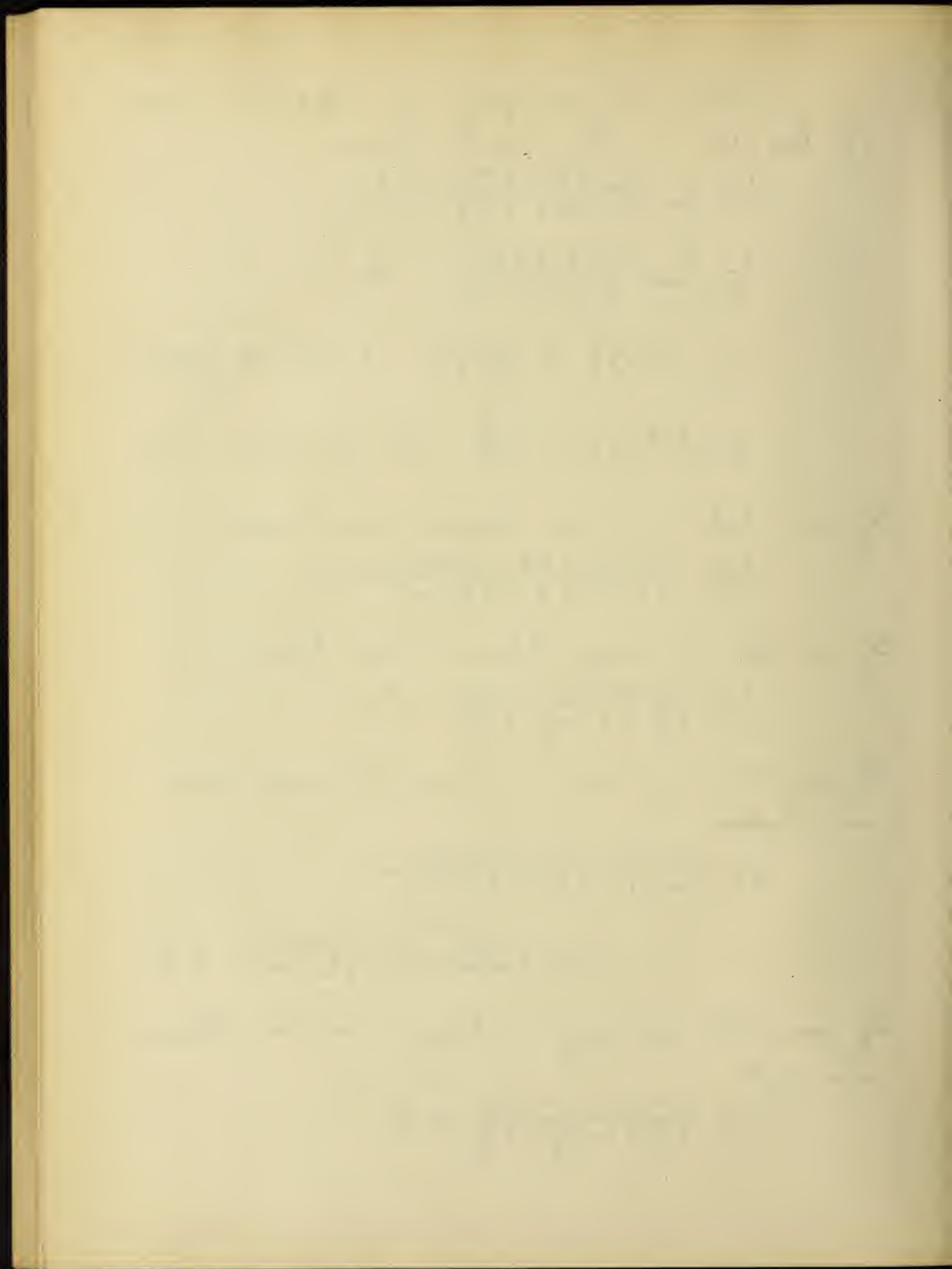
$$\lim_{x \neq 0} \frac{5m^2x^{2n} + 4x + 4mx^n}{3m^2x^{2n} + 2x + 2mx^n} =$$

$$\lim_{x \neq 0} \frac{5m^2x^{2n-1} + 4 + 4mx^{n-1}}{3m^2x^{2n-1} + 2 + 2mx^{n-1}} = 2$$

If we let $x = my^n$ where $1 < n < \infty$, then

we get

$$\lim_{y \neq 0} \frac{5y^2 + 4my^n + 4y}{3y^2 + 2my^n + 2y} = 2.$$



But if we let $y = mx^2 - x$, then we have

$$\begin{aligned} \lim_{x \neq 0} \frac{5(m^2x^4 - 2mx^3 + x^2) + 4x + 4(mx^2 - 4x)}{3(m^2x^4 - 2mx^3 + x^2) + 2x + 2(mx^2 - 4x)} \\ = \lim_{x \neq 0} \frac{5m^2x^2 - 10mx + 5 + 4m}{3m^2x^2 - 6mx + 3 + 2m} = \frac{5 + 4m}{3 + 2m} \end{aligned}$$

which shows that the double limit does not exist.

Example 14:- Given the function

$$z = \frac{(x+y)^4 + 2xy^2 + x^2y}{(x+y)^5 + xy^2 + x^2y}$$

to test for the existence of double limit at the origin.

Here we have a function in which the twice taken limits are ∞ while all limits obtained by linear approaches are finite. We have here

$$\lim_{x \neq 0} \lim_{y \neq 0} \frac{(x+y)^4 + 2xy^2 + x^2y}{(x+y)^5 + xy^2 + x^2y} = \lim_{x \neq 0} \frac{1}{x} = \infty,$$

$$\lim_{y \neq 0} \lim_{x \neq 0} f(x, y) = \lim_{y \neq 0} \frac{1}{y} = \infty,$$

$$\lim_{y \neq 0} f(\bar{x}, y) = \frac{1}{\bar{x}}, \text{ for constant } \bar{x} \neq 0,$$

$$\lim_{x \neq 0} f(x, \bar{y}) = \frac{1}{\bar{y}}, \text{ " " } \bar{y} \neq 0.$$

But if we put $y = mx$, then we have

$$\begin{aligned} \lim_{x \neq 0} \frac{x^4(1+m)^4 + 2m^2x^3 + mx^3}{x^5(1+m)^5 + m^2x^3 + mx^3} &= \lim_{x \neq 0} \frac{x(1+m)^4 + 2m^2 + m}{x^2(1+m)^4 + m^2 + m} \\ &= \frac{2m+1}{m+1} \end{aligned}$$

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which is always finite except for $m=0$ or ∞ . From this last equation we see by Prop. I. that the double limit does not exist.

Example 15:- Given the function

$$z = \frac{x^3 + 2xy + y^2}{x^2 + xy + y^2}$$

to find the limits.

In this function all the limits are finite and lie between 1 and 2 inclusive.

(1.) $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = 1.$

(2.) $\lim_{x \rightarrow 0} f(x, mx^\mu) = 1$, for $\infty > \mu > 2$, m arbitrary.

(3.) $\lim_{x \rightarrow 0} f(x, mx^\mu) = \frac{1+2m}{1+m}$, " $\mu=2$ and for $0 < m < \infty$,
we have $1 < \frac{1+2m}{1+m} < 2$.

(4.) $\lim_{x \rightarrow 0} f(x, mx^\mu) = \frac{2+m}{1+m}$, for $\mu=1$ and for $0 < m < \infty$,
we have $2 > \frac{2+m}{1+m} > 1$.

(5.) $\lim_{x \rightarrow 0} f(x, mx^\mu) = 2$, for $2 > \mu > 1$, m arbitrary.

(6.) $\lim_{x \rightarrow 0} f(x, mx^\mu) = 1$, for $0 < \mu < 1$, m "

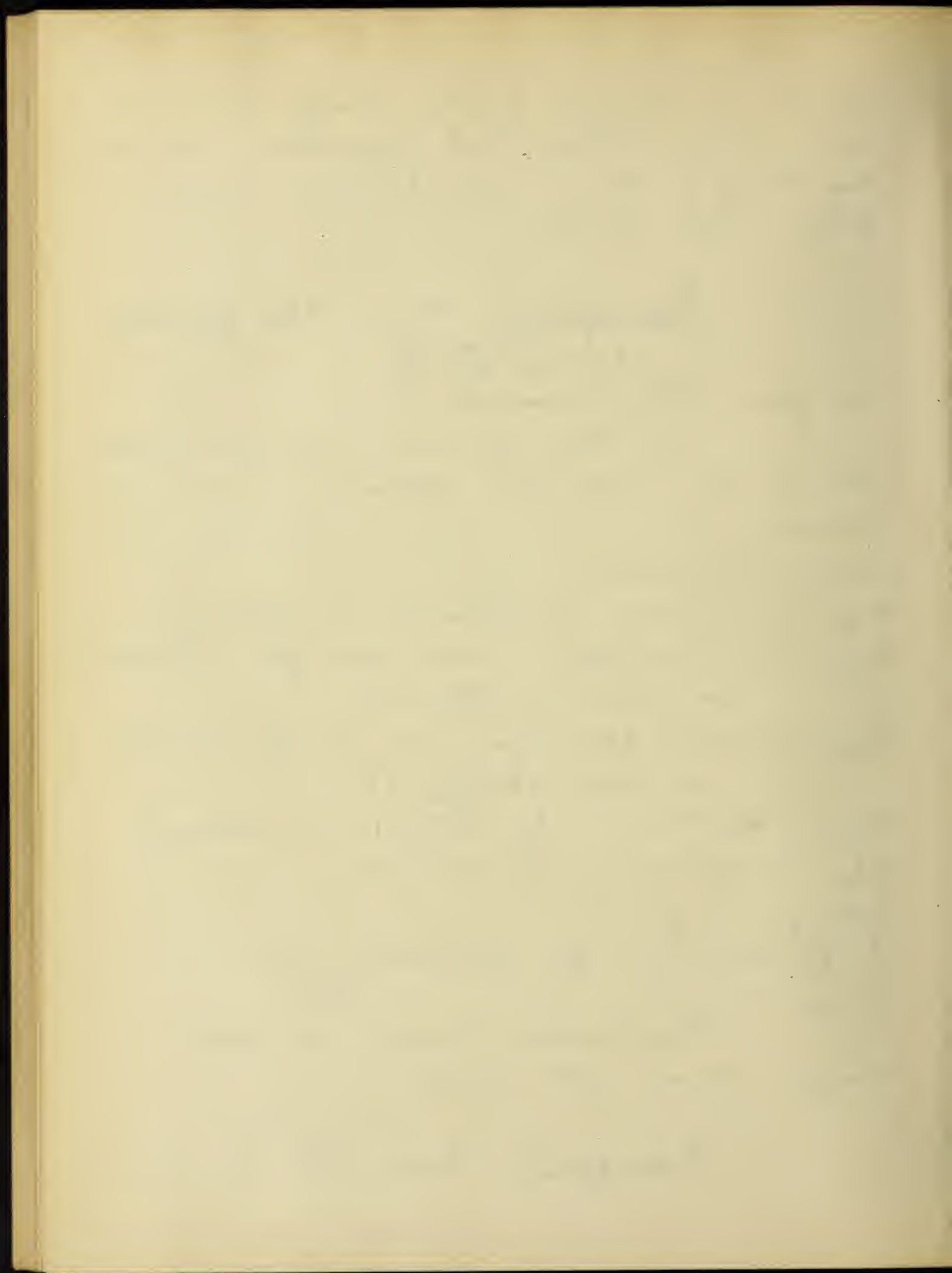
(7.) $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = 1.$

(8.) $\lim_{x \rightarrow 0} f(x, \bar{y}) = 1$, for constant $y \neq 0$.

(9.) $\lim_{y \rightarrow 0} f(\bar{x}, y) = 1$, " " $x \neq 0$.

The double limit at the origin does not exist.

Example 16:- Given the function



$z = \frac{a(x+y)^4 + b x^2 y + c x y^2}{a_1(x+y)^4 + b_1 x^2 y + c_1 x y^2}$; where $f(0,0) = \frac{a}{a_1}$,
to test for continuity and for double
limit at the origin.

This function is continuous at
the point $(0,0)$ with respect to x alone
and with respect to y alone.

$\lim_{x \neq 0} \lim_{y \neq 0} \frac{a(x+y)^4 + b x^2 y + c x y^2}{a_1(x+y)^4 + b_1 x^2 y + c_1 x y^2} = \frac{a}{a_1} = f(0,0)$
therefore it is continuous with respect to
 x alone.

$\lim_{y \neq 0} \lim_{x \neq 0} f(x,y) = \frac{a}{a_1} = f(0,0)$
therefore it is continuous with respect
to y alone. Moreover we have

$$\lim_{x \neq 0} f(x, \bar{y}) = \frac{a}{a_1}, \text{ for constant } y \neq 0,$$

$$\lim_{y \neq 0} f(\bar{x}, y) = \frac{a}{a_1}, \text{ " " " } x \neq 0.$$

Still the double limit does not exist;

for, let $y = mx$ and we get

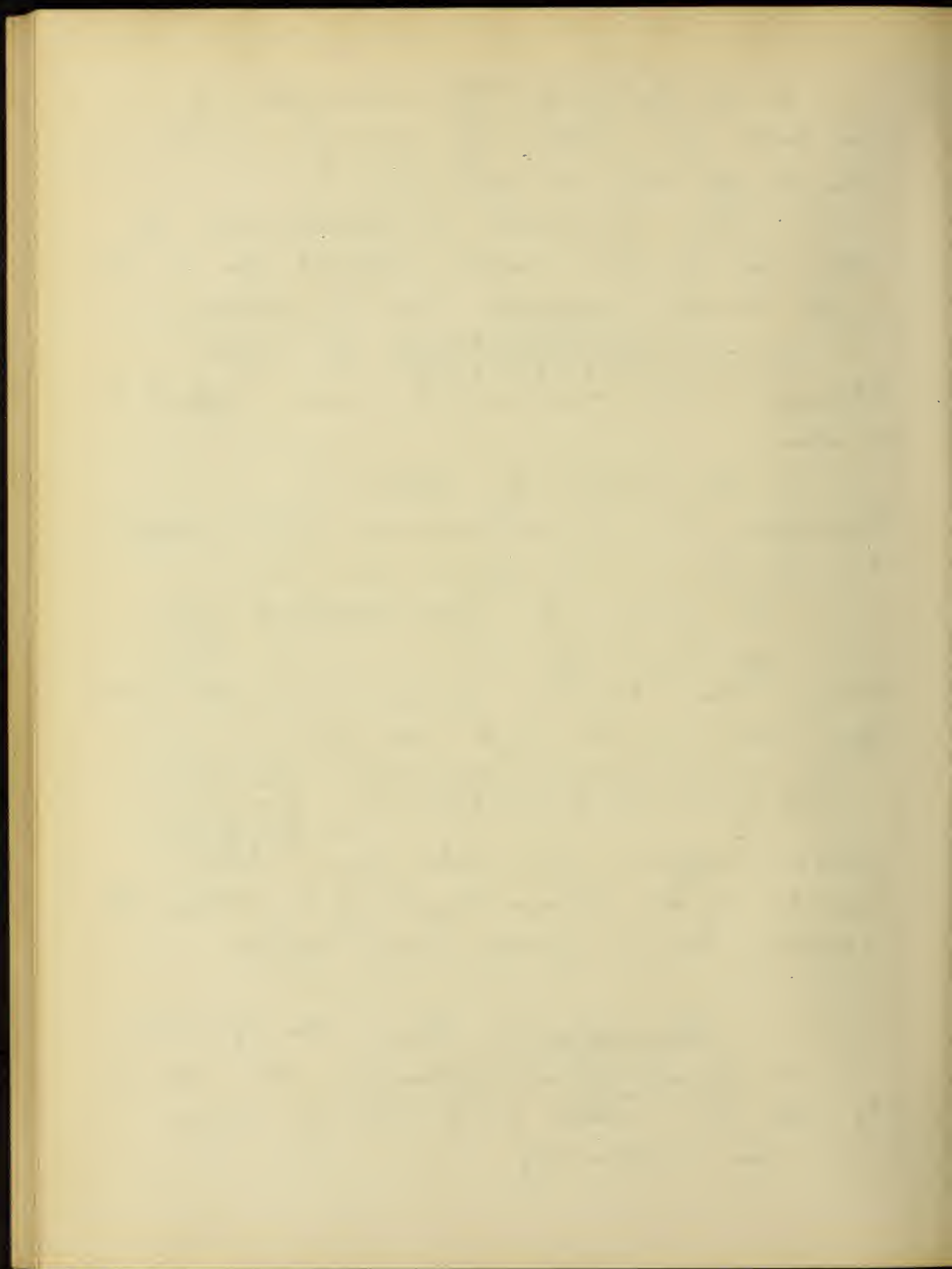
$$\lim_{x \neq 0} \frac{a x^4 (1+m)^4 + b m x^3 + c m^2 x^3}{a_1 x^4 (1+m)^4 + b_1 m x^3 + c_1 m^2 x^3} = \frac{b m + c m^2}{b_1 m + c_1 m^2} = \frac{b + c m}{b_1 + c_1 m}$$

which differs from the twice taken
limits and shows, by Prop. I, that the
double limit does not exist.

Example 17:- Given the function

$$z = \frac{6(x+y)^4 + 2y^2(y-x) + 6y(y-x)^2}{3(x+y)^4 + 5y^2(y-x) + 3y(y-x)^2}; \text{ where } f(0,0) = 2.$$

to test for double limit at the origin
and for continuity.



If we let $y = x + mx^2$, then we have

$$\lim_{x \neq 0} \frac{6(2x + mx^2)^4 + 2(x + mx^2)^2 mx^2 + 6(x + mx^2) m^2 x^4}{3(2x + mx^2)^4 + 5(x + mx^2)^2 mx^2 + 3(x + mx^2) m^2 x^4}$$

$$= \frac{6 \cdot 2^4 + 2m + 6m^2}{3 \cdot 2^4 + 5m + 3m^2}$$

Therefore the double limit does not exist, by Prop. I, since this may have different values for different values of m .

$$\lim_{x \neq 0} \lim_{y \neq 0} f(x, y) = 2 = f(0, 0)$$

$$\lim_{y \neq 0} \lim_{x \neq 0} f(x, y) = \lim_{y \neq 0} \frac{6y^4 + 2y^3 + 6y^2}{3y^4 + 5y^3 + 3y^2} = 1$$

Therefore it is continuous with respect to x alone, but not continuous with respect to y alone.

If we let $y = mx^2$, then we have

$$\lim_{x \neq 0} \frac{6x^4(1+mx)^4 + 2m^2x^5(mx-1) + 6mx^4(mx-1)^2}{3x^4(1+mx)^4 + 5m^2x^5(mx-1) + 3mx^4(mx-1)^2}$$

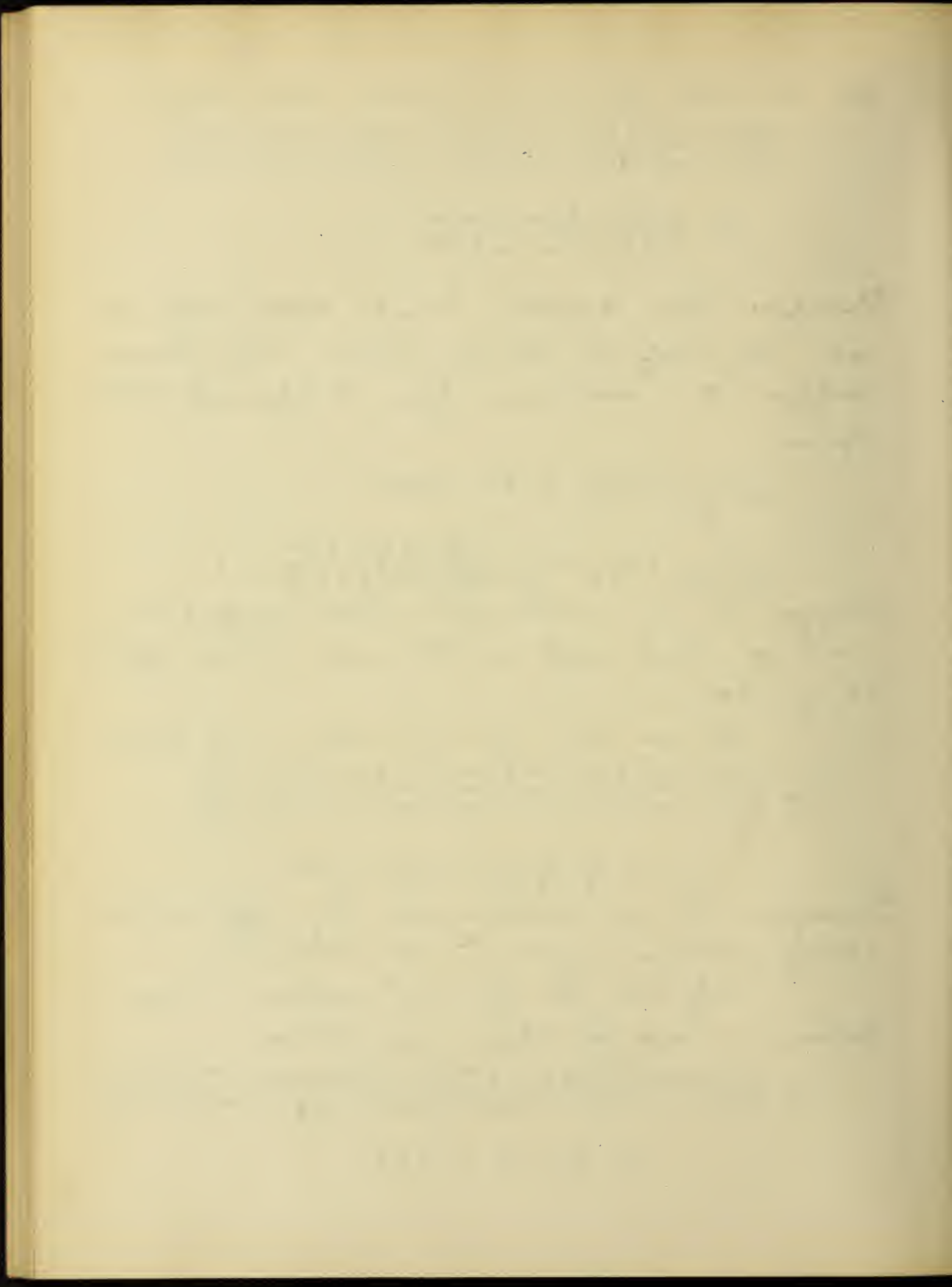
$$= \frac{6-6m}{3-3m} = 2 = f(0, 0)$$

Therefore it is continuous by approaches along curves $y = mx^2$, m arbitrary.

If we let $y = mx^\mu$, where μ lies between 2 and ∞ , then we have

$$\lim_{x \neq 0} \frac{6x^4(1+mx^{\mu-1})^4 + 2m^2x^{2\mu+1}(mx^{\mu-1}-1) + 6mx^{\mu+2}(mx^{\mu-1}-1)^2}{3x^4(1+mx^{\mu-1})^4 + 5m^2x^{2\mu+1}(mx^{\mu-1}-1) + 3mx^{\mu+2}(mx^{\mu-1}-1)^2}$$

$$= \frac{6}{3} = 2 = f(0, 0)$$



Therefore it is continuous by approaches along curves $y = mx^\mu$, where $2 < \mu < \infty$.

Example 18:- Given the function

$$z = \frac{9x^7 + y^2(y-x^2) + 9y(y-x^2)^2}{3x^7 + 2y^2(y-x^2) + 3y(y-x^2)^2}$$

to test for the existence of the double limit at the origin.

Here we have a function in which we get the same limiting values by all approaches of form $y = mx^\mu$ except the linear and the quadratic approaches.

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \frac{9x^7}{3x^7} = 3.$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \frac{y^3 + 9y^3}{2y^3 + 3y^3} = 2.$$

Therefore the double limit does not exist at the origin, since the twice taken limits are not equal.

$$\lim_{y \rightarrow 0} f(\bar{x}, y) = 3, \text{ for constant } \bar{x} \neq 0.$$

$$\lim_{x \rightarrow 0} f(x, \bar{y}) = 2, \quad " \quad " \quad y \neq 0.$$

If we put $y = mx$, then we have

$$\lim_{x \rightarrow 0} \frac{9x^7 + m^2x^3(m-x) + 9mx^3(m-x)^2}{3x^7 + 2m^2x^3(m-x) + 3mx^3(m-x)^2} = \frac{m^3 + 9m^3}{2m^3 + 3m^3} = 2$$

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If we put $y = mx^2$, then we get

$$\lim_{x \neq 0} \frac{9x^7 + m^2 x^6(m-1) + 9m x^6(m-1)^2}{3x^7 + 2m^2 x^6(m-1) + 3m x^6(m-1)^2} = \frac{m^2 + 9m(m-1)}{2m^2 + 3m(m-1)}$$

$$= \frac{10m - 9}{5m - 3}$$

If we put $y = mx^\mu$ where $2 < \mu < \infty$, then we have

$$\lim_{x \neq 0} \frac{9x^7 + m^2 x^{2\mu+2}(mx^{\mu-2}-1) + 9m x^{\mu+4}(mx^{\mu-2}-1)^2}{3x^7 + 2m^2 x^{2\mu+2}(mx^{\mu-2}-1) + 3m x^{\mu+4}(mx^{\mu-2}-1)^2}$$

$$= \frac{9 - 9m}{3 - 3m} = 3, \text{ for } \mu = 3$$

$$= \frac{9}{3} = 3, \text{ " } \mu = 4, 5, \dots \infty.$$

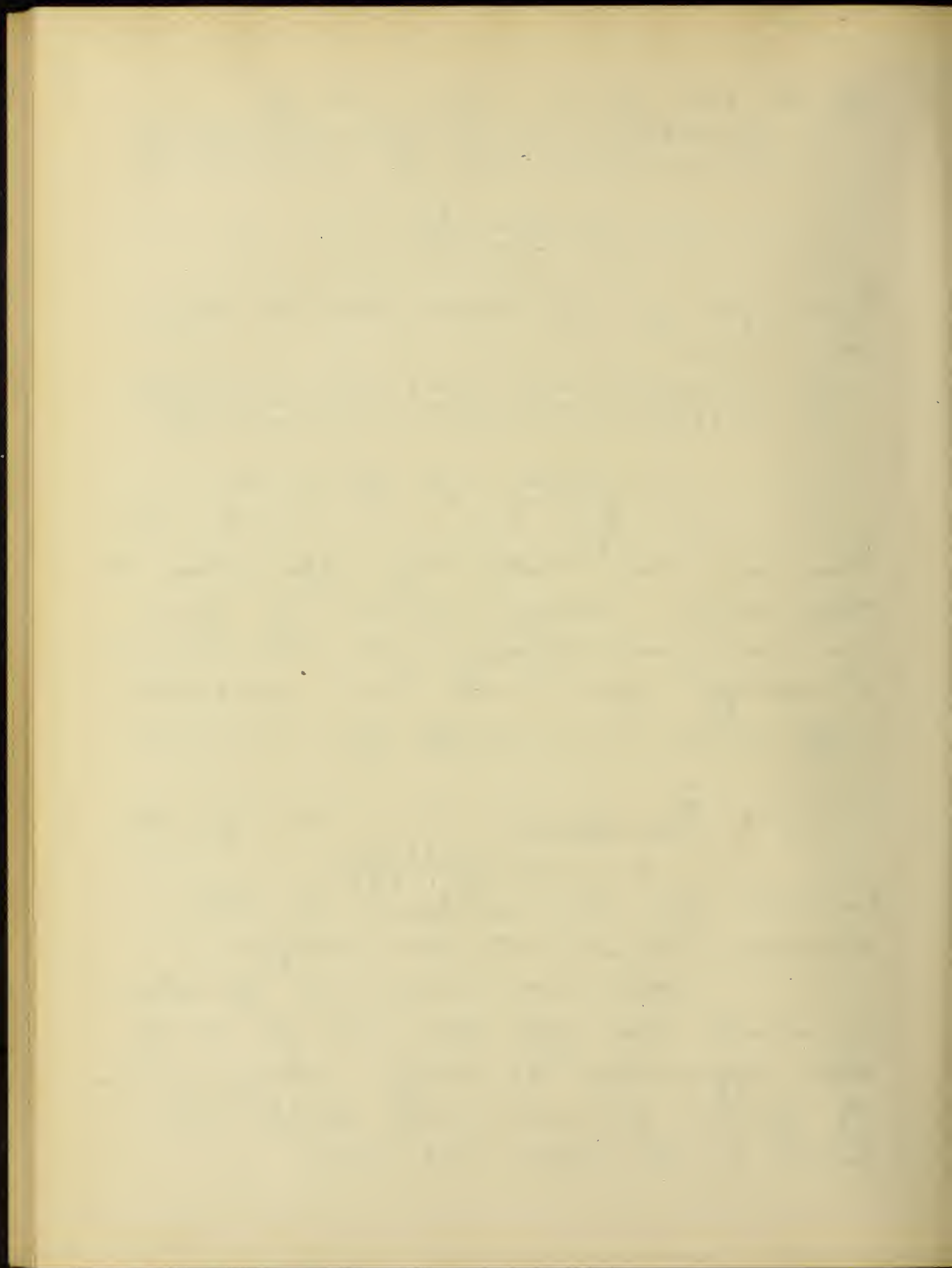
Thus we see that by approaches to the origin along curves of form $y = mx^\mu$, we always get the limit 3, except for linear and quadratic approaches, i.e. except for $\mu = 1$, or 2.

Example 19:- Given the function

$$Z = (x+y) \frac{y + (x+y)^2}{y - (x+y)^2}$$

to test for the existence of the double limit at the origin.

Here we have a function in which we get the limit 0 by all approaches to origin along curves of form $y = mx^\mu$, and still the double limit does not exist.



$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} (x+y) \frac{y+(x+y)^2}{y-(x+y)^2} = 0.$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} y \frac{1+y}{1-y} = 0.$$

$$\lim_{x \rightarrow 0} f(x, \bar{y}) = \bar{y} \frac{1+\bar{y}}{1-\bar{y}}, \text{ for constant } y \neq 0.$$

$$\lim_{y \rightarrow 0} f(\bar{x}, y) = -\bar{x}, \quad \text{"} \quad \text{"} \quad x \neq 0.$$

If we let $y = mx$, then we get

$$\lim_{x \rightarrow 0} x(1+m) \frac{mx + x^2(1+m)^2}{mx - x^2(1+m)^2} = 0.$$

If we let $y = mx^\mu$, where $0 < \mu < \infty$, we get

$$\lim_{x \rightarrow 0} x(1+mx^{\mu-1}) \frac{mx^{\mu-2} + (1+mx^{\mu-1})^2}{mx^{\mu-2} - (1+mx^{\mu-1})^2} = 0.$$

But if we let $y = x^2 + mx^3$, then we get

$$\lim_{x \rightarrow 0} (x+x^2+mx^3) \frac{(1+mx) + (1+x+mx^2)^2}{(1+mx) - (1+x+mx^2)^2}$$

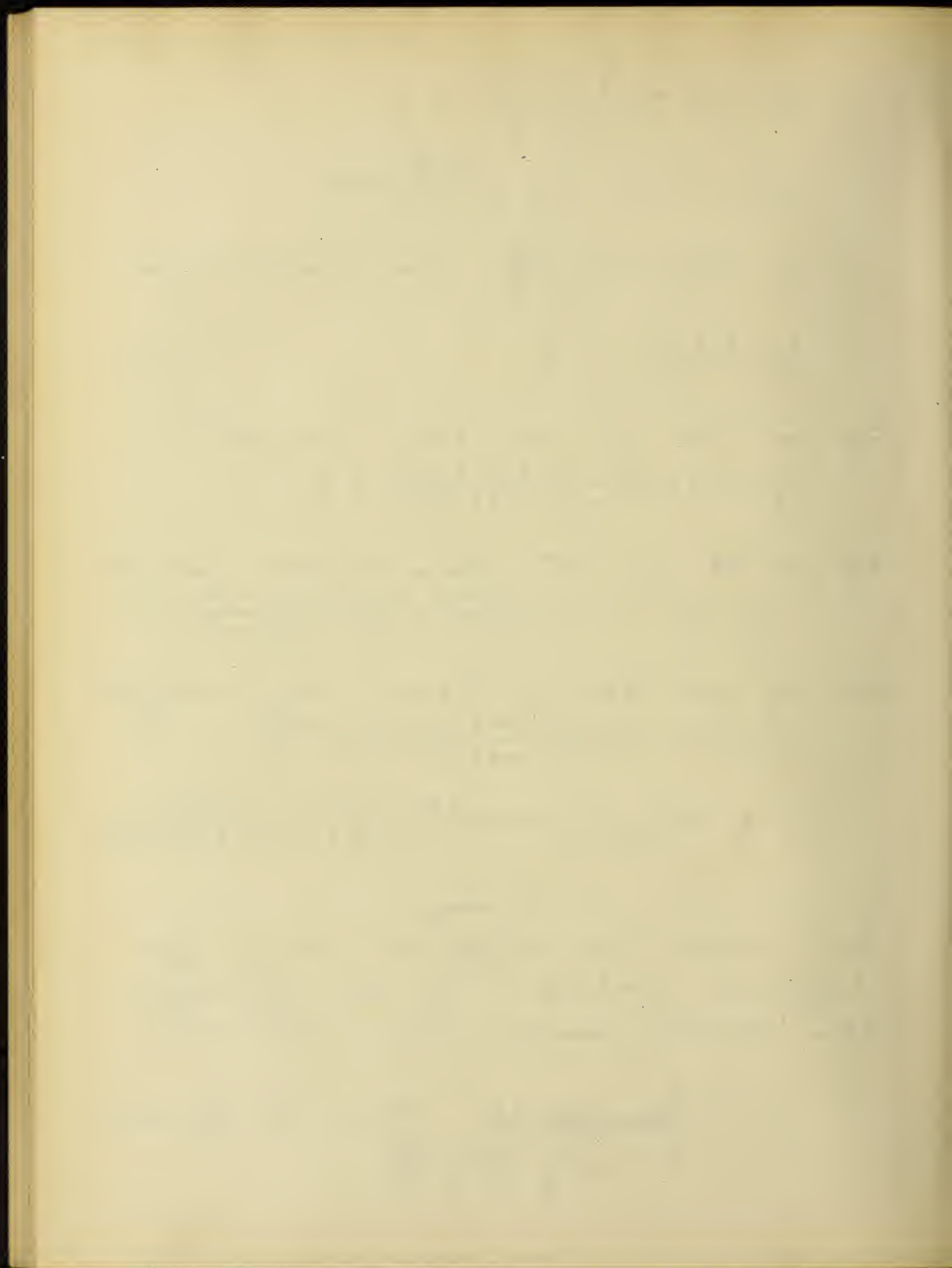
$$= \lim_{x \rightarrow 0} (1+x+mx^2) \frac{(1+mx) + (1+x+mx^2)^2}{m-2 + (1+2m)x^2 + 2mx^3 + m^2x^4}$$

$$= \frac{1}{m-2}.$$

Since this has different values for different values of m , by Prop. I, the double limit does not exist.

Example 20:- Given the function

$$Z = \frac{1}{x+y} \frac{y-(x+y)^2}{y+(x+y)^2}$$



to test for the existence of the double limit at the origin.

Here we have a function in which we get the limiting value ∞ by all approaches along curves of form $y = mx^\mu$, where $0 < \mu < \infty$.

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{1}{x+y} \frac{y - (x+y)^2}{y + (x+y)^2} = -\infty.$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \frac{1}{y} \cdot \frac{1-y}{1+y} = \infty.$$

$$\lim_{x \rightarrow 0} \frac{1}{x+y} \frac{y - (x+y)^2}{y + (x+y)^2} = \frac{1}{y} \frac{1-y}{1+y}, \text{ for constant } y \neq 0$$

$$\lim_{y \rightarrow 0} f(\bar{x}, y) = -\frac{1}{\bar{x}}, \text{ for constant } \bar{x} \neq 0.$$

If we let $y = mx^\mu$, where $0 < \mu < \infty$, we get

$$\lim_{x \rightarrow 0} \frac{1}{x(1+mx^{\mu-1})} \frac{mx^{\mu-2} - (1+mx^{\mu-1})^2}{mx^{\mu-2} + (1+mx^{\mu-1})^2} = -\infty.$$

If we put $y = x^2 + mx^3$, we get

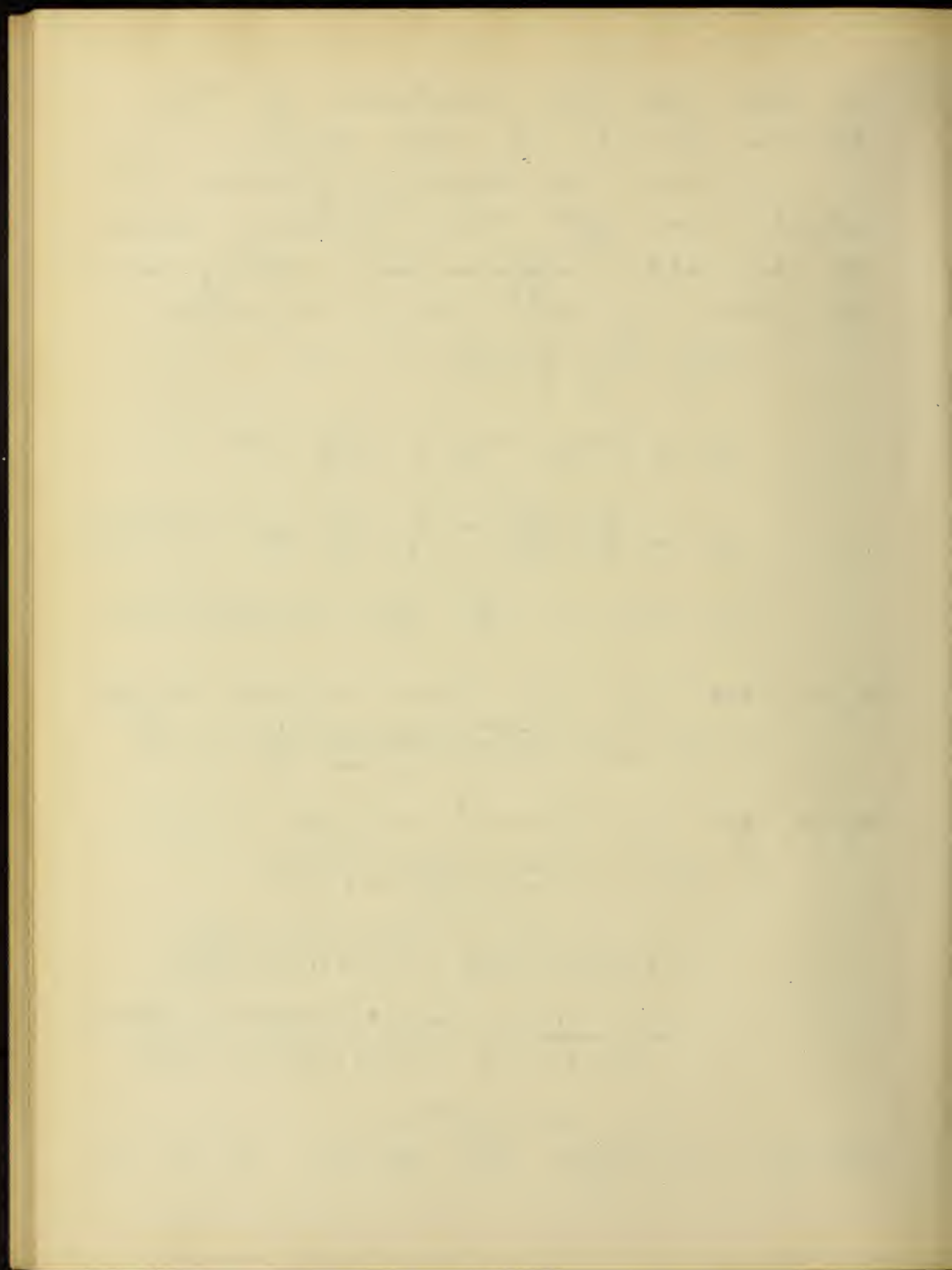
$$\lim_{x \rightarrow 0} \frac{1}{x + x^2 + mx^3} \frac{x^2 + mx^3 - (x + x^2 + mx^3)^2}{x^2 + mx^3 + (x + x^2 + mx^3)^2}$$

$$= \lim_{x \rightarrow 0} \frac{1}{x(1+x+mx^2)} \frac{1+mx - (1+x+mx^2)^2}{1+mx + (1+x+mx^2)^2}$$

$$= \frac{1}{x(1+x+mx^2)} \frac{(m-2)x - (1+2m)x^2 - (2mx^3 + m^2x^4)}{1+mx + (1+x+mx^2)^2}$$

$$= \frac{m-2}{2}.$$

Therefore, by Prop I, the double limit does



not exist.

Example 21:- Given the function

$$z = x \frac{x+y}{x^2-y}$$

to test for the existence of the double limit at the origin.

In this function all the single limits exist but the double limit does not exist.

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} x \frac{x+y}{x^2-y} = 1.$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = 0.$$

$$\lim_{x \rightarrow 0} f(x, \bar{y}) = 0, \text{ for constant } y \neq 0.$$

$$\lim_{y \rightarrow 0} f(\bar{x}, y) = 1, \quad " \quad " \quad x \neq 0.$$

If we put $y = mx$, then we get

$$\lim_{x \rightarrow 0} x \frac{x+mx}{x^2-mx} = \lim_{x \rightarrow 0} \frac{1+m}{1-\frac{m}{x}} = 0.$$

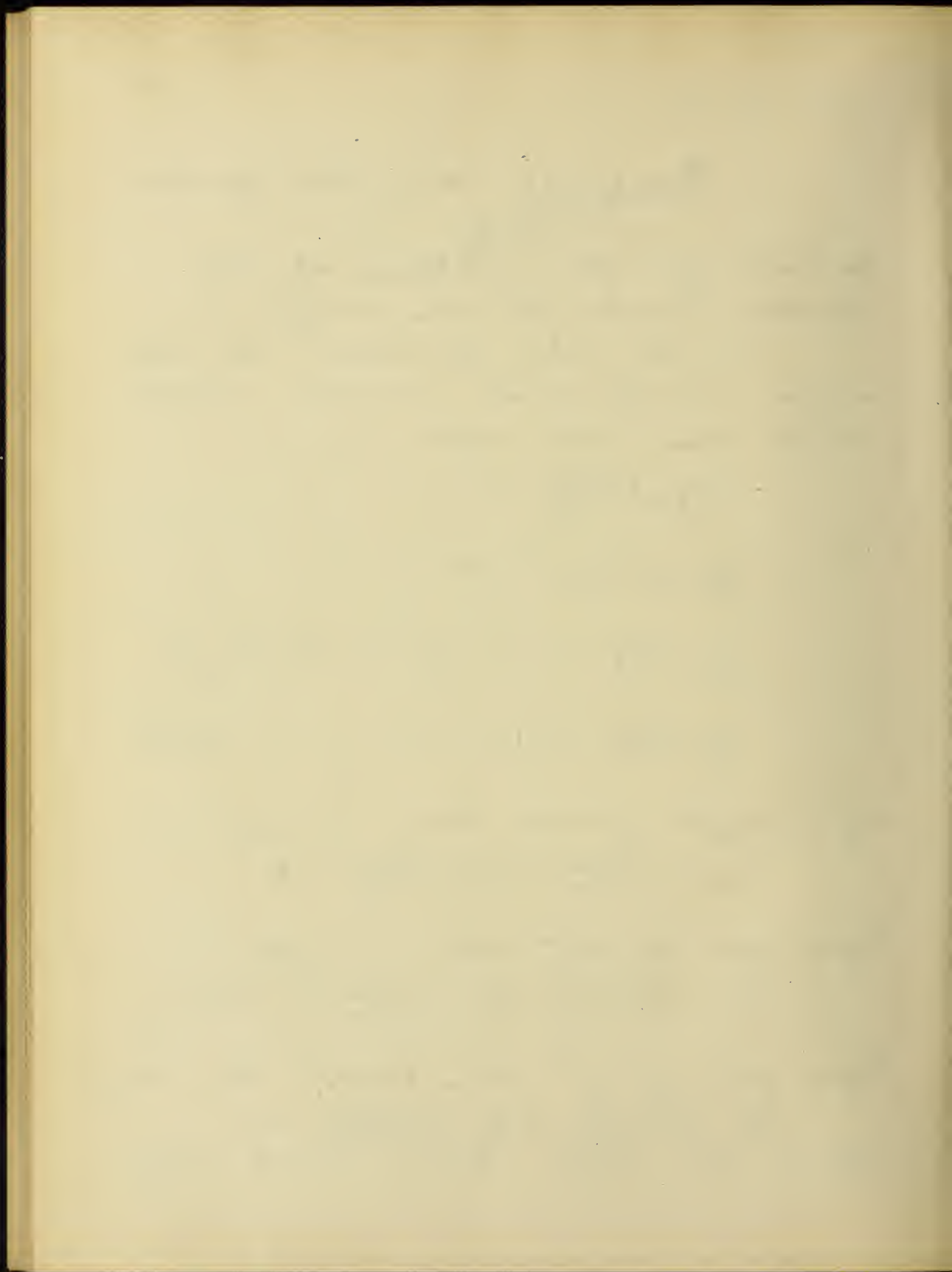
If we put $y = mx^2$, then we get

$$\lim_{x \rightarrow 0} x \frac{x+mx^2}{x^2-mx^2} = \lim_{x \rightarrow 0} \frac{1+mx}{1-m} = \frac{1}{1-m}.$$

If we put $y = mx^\mu$, where $2 < \mu < \infty$, then we get

$$\lim_{x \rightarrow 0} x \frac{x+mx^\mu}{x^2-mx^\mu} = \lim_{x \rightarrow 0} \frac{1+mx^{\mu-1}}{1-mx^{\mu-2}} = 1.$$

Since the twice taken limits differ



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we know the double limit does not exist by Prop. II.

Example 22:- Given the function

$$Z = \log \frac{y}{x}$$

to test for the existence of the double limit at the origin.

Here we have

$$\lim_{x \neq 0} \lim_{y \neq 0} \log \frac{y}{x} = -\infty.$$

$$\lim_{y \neq 0} \lim_{x \neq 0} \log \frac{y}{x} = \infty.$$

$$\lim_{x \neq 0} \log \frac{y}{x} = -\infty, \text{ for constant } y \neq 0.$$

$$\lim_{y \neq 0} \log \frac{y}{x} = +\infty, \quad " \quad " \quad x \neq 0.$$

If we let $y = e^m x$, then we get

$$\lim_{x \neq 0} \log \frac{e^m x}{x} = \log e^m = m.$$

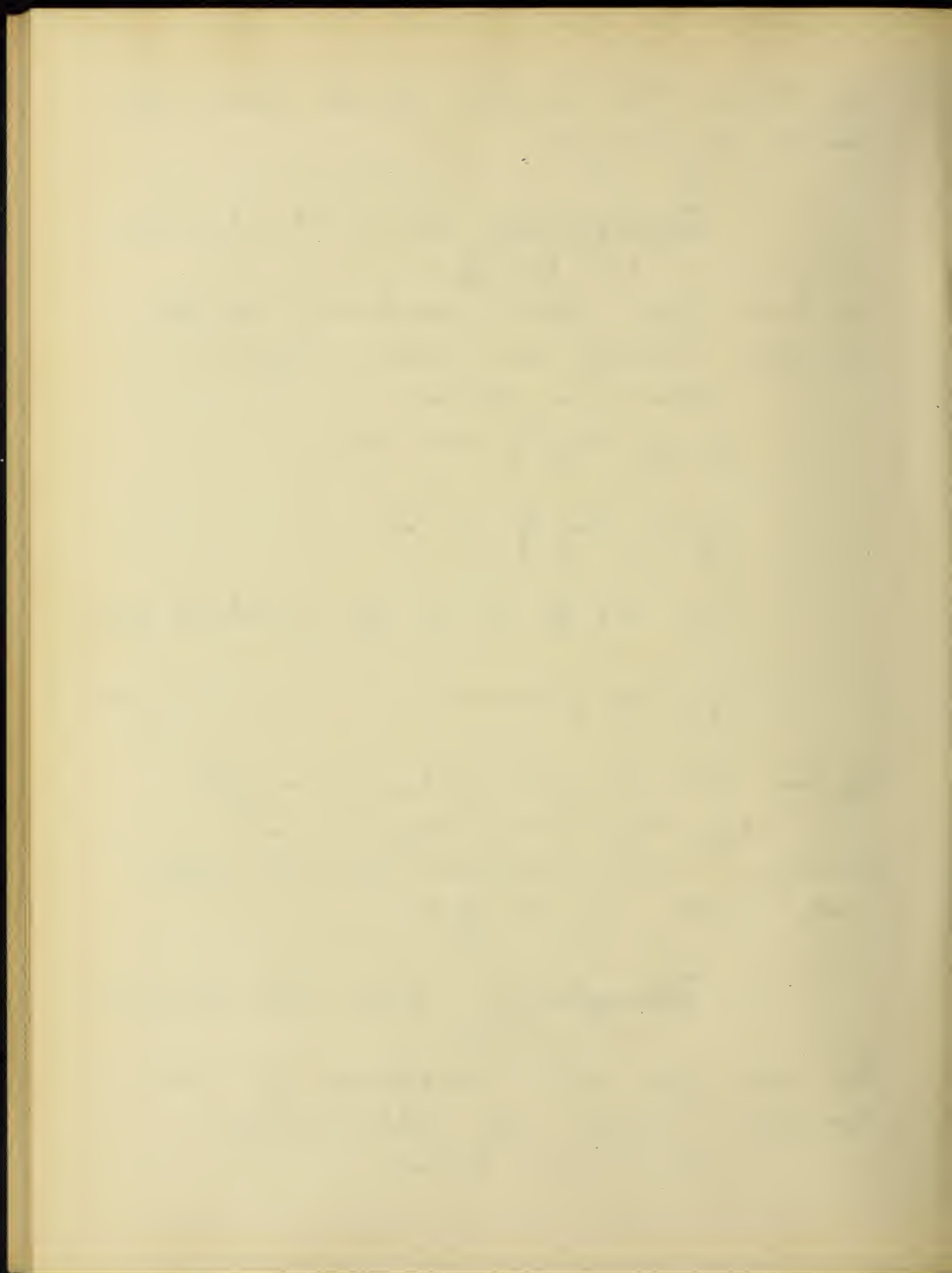
Therefore the double limit does not exist by Prop. I.

Example 23:- Given the function

$$Z = y \log x$$

to test for the existence of the double limit at the origin.

Here we have



$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} y \log x = 0.$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} y \log x = -\infty.$$

Since these twice taken limits are not equal we know, by Prop. II, that the double limit at the origin does not exist.

We should notice that while

$$\lim_{x \rightarrow 0} x \log x = 0$$

yet the double limit

$$\lim_{\substack{y \rightarrow 0 \\ x \rightarrow 0}} y \log x \neq 0$$

and in fact does not exist at all.

Example 24:- Given the function

$$Z = a x^{\mu_1} y^{\mu_2}$$

to test for the existence of the double limit at the origin.

If the μ 's are integral and positive we have here a general integral rational function of a single term.

If we put $y = x$, then we get

$$\lim_{x \rightarrow 0} a x^{\mu_1 + \mu_2} = 0.$$

Therefore 0 must be the value of the double limit if there be one. Substituting 0 in the defining relation,

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My dear Mr. Garrison

I have just received your letter of the 28th inst. and am
glad to hear that you are so interested in the
cause of the colored people. I am sure that your
efforts will be successful in securing for them
the same rights and privileges as are enjoyed by
the white race.

I am, Sir, very respectfully,
Your obedient servant,
Wm. L. Chapin

(Chap. I § 2.) we get

$$|a(0+\delta_1)^{\mu_1}(0+\delta_2)^{\mu_2} - 0| < \sigma$$

or

$$|a \delta_1^{\mu_1} \delta_2^{\mu_2}| < \sigma$$

which relation is satisfied. Therefore the double limit exists and is equal to 0. By Prop. VIII we know that the sum or difference of a finite number of such terms as above has the double limit 0. Therefore every integral rational function of x and y with finite number of terms has the double limit 0.

